# MATH 4120, Finding Roots of Polynomials

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## 1 Linear Polynomials

The basic linear polynomial equation looks like

$$ax + b = 0$$

where $a$ and $b$ are given constants with $a \neq 0$. We are asked to find a value for $x$ that makes the equation hold. Such $x$ is called a *root* of the equation. If we allow that we can divide numbers and take their negations, then the linear equation can be converted into

$$ax = -b \quad \text{and so} \quad x = -\frac{b}{a}$$

So if $3x + 2 = 0$, then $x = -2/3$. Although this equation only employs natural numbers like 0, 2, and 3 from $\mathbb{N}$, the solution is a rational number $(-2/3) \in \mathbb{Q}$ which is not even an integer in $\mathbb{Z}$. We consider the rational numbers a proper environment in which our equations and solutions live. Note that basic linear equations have exactly one solution.

## 2 Quadratic Polynomials

The basic quadratic polynomial equation looks like

$$ax^2 + bx + c = 0$$

where $a$, $b$, and $c$ are given constants with $a \neq 0$. Finding a root for the quadratic equation is a lot harder than for the linear equation. Since we accept fractions, the equation can be rewritten as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

When we write $d$ for $b/a$ and $e$ for $c/a$, we get

$$x^2 + dx + e = 0$$

If we solve this equation, then we have a solution for the original quadratic equation. We say that we may suppose without loss of generality that the leading coefficient (the $a$ in the original polynomial above) equals 1. We continue working with the new and simpler quadratic polynomial. Some people came up with the following trick. Replace $x$ by $y - t$, where $y$ is also a variable, but $t$ is supposed to be a cleverly chosen constant. Which constant we don't know until we substitute and get further insights. Substitution of $x = y - t$ and some algebra give

$$(y - t)^2 + d(y - t) + e = 0$$

$$y^2 - 2yt + t^2 + dy - dt + e = 0$$

$$y^2 + (d - 2t)y + t^2 - dt + e = 0$$

Now we pick for $t$ a constant value such that $d - 2t = 0$, that is, set $t = d/2$. So $x = y - (d/2)$ and

$$y^2 + \frac{d^2}{4} - \frac{d^2}{2} + e = 0 \quad \text{and so} \quad y^2 = \frac{d^2}{4} - e$$

We have the notation of square roots for solutions of such equations. So, with the $\pm$ abbreviation,

$$y = \pm\sqrt{\frac{d^2}{4} - e}$$

are solutions, usually two different ones. In terms of $x = y - (d/2)$ we get

$$x = -\frac{d}{2} \pm \sqrt{\frac{d^2}{4} - e} \quad \text{or equivalently} \quad x = \frac{-d \pm \sqrt{d^2 - 4e}}{2}$$

If you wish, you can replace $d$ by $b/a$ and $e$ by $c/a$. Here we prefer to stay with $d$ and $e$. Square roots need not be rational numbers. For example $x^2 - x - 1 = 0$ has solutions $x = (1 + \sqrt{5})/2$ and $x = (1 - \sqrt{5})/2$, neither of which is in $\mathbb{Q}$. They are real numbers, so $\mathbb{R}$ is another proper environment in which our equations and solutions may live.

Some illustrative examples:

Given $x^2 - 12x + 35 = 0$, we get

$$x = \frac{12 \pm \sqrt{144 - 140}}{2}$$

So $x = 6 \pm 1$, and we have solutions $x = 7$ and $x = 5$. Note that we have $x^2 - 12x + 35 = (x-5)(x-7)$.

Given $x^2 - 12x + 36 = 0$, we get

$$x = \frac{12 \pm \sqrt{144 - 144}}{2}$$

So $x = 6 \pm 0$, and we have solution $x = 6$. Note that we have $x^2 - 12x + 36 = (x-6)(x-6) = (x-6)^2$. Although there is only one solution, this final equation suggests why 6 is called a double root of equation $x^2 - 12x + 36 = 0$.

Given $x^2 - 12x + 37 = 0$, we get

$$x = \frac{12 \pm \sqrt{144 - 148}}{2}$$

So $x = 6 \pm \sqrt{-1}$, and we have solutions $x = 6 + i$ and $x = 6 - i$, where $i$ is the common abbreviation for $\sqrt{-1}$. Note that we have $x^2 - 12x + 37 = (x - 6 - i)(x - 6 + i)$. This example indicates that the complex plane $\mathbb{C}$ is another proper environment in which our equations and solutions may live.

## 3 Cubic Polynomials

The basic cubic polynomial equation (with leading coefficient 1) looks like

$$x^3 + ax^2 + bx + c = 0$$

where $a$, $b$, and $c$ are given constants. Find $x$. Because of our experience with quadratic polynomials, we already have set the leading coefficient equal to 1. Further experience with quadratic equations suggests we replace $x$ by $y - t$, where $y$ is also a variable, but $t$ is a cleverly chosen constant. Which constant to pick we only discover after substitution. Without repeating all the calculations, we state that $t = a/3$ is our choice. So $x = y - a/3$ and

$$y^3 + (b - \frac{a^2}{3})y + (\frac{2a^3}{27} - \frac{ab}{3} + c) = 0$$

When we write $p$ for $b - a^2/3$ and $q$ for $2a^3/27 - ab/3 + c$, we get

$$y^3 + py + q = 0$$

Now what? Some people came up with the following trick. Replace $y$ by $w + s/w$, where $w$ is also a variable, and $s$ is a cleverly chosen constant. Which constant we don't know until we perform the substitution. We now return to the book, where we read that $s = -p/3$ is a good choice.