Complex exponentiation

From calculus over the reals $\mathbb{R}$ we know that the standard exponential function equals the limit of the power series function

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots = \sum_{n \geq 0} \frac{x^n}{n!}$$

This equation holds for all real numbers x. The basic properties about convergence of power series also work for complex numbers. One easily shows that the complex number power series

$$1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \ldots = \sum_{n \geq 0} \frac{z^n}{n!}$$

converges to a (complex number) limit, for all complex values for z. We *define* the expression e^z as name for this limit. So

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \ldots = \sum_{n \geq 0} \frac{z^n}{n!}$$

The complex number function e^z still satisfies many nice properties that we know, like

$$e^0 = 1$$
$$e^{z+w} = e^z e^w$$

Here is an almost magical fact: One can show that for all real numbers a and b,

$$e^{a+ib} = e^a \cos(b) + ie^a \sin(b)$$

So, in particular,

$$\cos(b+d) + i\sin(b+d) = e^{i(b+d)} =$$
$$e^{ib}e^{id} = (\cos(b) + i\sin(b))(\cos(d) + i\sin(d)) =$$
$$(\cos(b)\cos(d) - \sin(b)\sin(d)) + i(\cos(b)\sin(d) + \sin(b)\cos(d))$$

and presto, out come the trigonometric formulas

$$\cos(b+d) = \cos(b)\cos(d) - \sin(b)\sin(d)$$
$$\sin(b+d) = \cos(b)\sin(d) + \sin(b)\cos(d)$$

Finally, let us *define* complex differentiation in steps. First define

$$\frac{d\,1}{dz} = 0 \quad \text{and} \quad \frac{d\,z^n}{dz} = nz^{n-1} \text{ for all } n > 0$$

Extend this definition to all power series in the natural way, through sums, constant multiples, and uniform limits. The resulting complex derivative still satisfies the product rule and the chain rule. We leave it as an exercise to show through term by term differentiation of the power series above, that

$$\frac{d\,e^{\lambda z}}{dz} = \lambda e^{\lambda z}$$

for all constants λ.