Exercise 1.
Find the **global** maxima and minima of the function $f(x, y) = 2x^2 + y^2$ given the constraint $x^2 + y^2 \leq 1$.

A solution.
First the internal potential extrema, where $x^2 + y^2 < 1$. This only requires finding the critical points, where the function f is horizontal. So compute

$$\text{grad} f = \nabla f = (f_x, f_y) = (4x, 2y).$$

Critical points are where $\nabla f = (4x, 2y) = (0, 0)$. So we have the two equations

$$\begin{aligned} &4x = 0 \quad \text{and} \\ &2y = 0 \end{aligned}$$

So $(0, 0)$ is the only critical point and it satisfies $x^2 + y^2 < 1$ and $f(0, 0) = 0$. So we have candidate:

$f(0, 0) = 0$ at $(0, 0)$ (call it **candidate A**)

Second, the boundary potential extrema, where $g(x, y) = x^2 + y^2 = 1$. This requires Lagrangian multipliers, where the 'steepest incline' of f on the curve $g = 1$ is parallel to the 'steepest incline' of g. So we get equations

$$\begin{aligned} &\nabla f = \lambda \nabla g \quad \text{and} \\ &g = 1 \end{aligned}$$

More explicitly,

$$\begin{aligned} &(4x, 2y) = \lambda(2x, 2y) \quad \text{and} \\ &x^2 + y^2 = 1 \end{aligned}$$

So solve the three equations in three unknowns:

1. $4x = \lambda 2x$ equivalently $2x(2 - \lambda) = 0$
2. $2y = \lambda 2y$ equivalently $2y(1 - \lambda) = 0$
3. $x^2 + y^2 = 1$

Equation (1), try $\lambda = 2$. Then equation (2) gives $y = 0$, and then equation (3) gives $x = \pm 1$. So we have candidates:

$f(1, 0) = 2$ at $(1, 0)$ (call it **candidate B**)
$f(-1, 0) = 2$ at $(-1, 0)$ (call it **candidate C**)

Finally, try equation (1) with $\lambda \neq 2$. Then this same equation (1) gives $x = 0$, and then equation (3) gives $y = \pm 1$. So we may have points $(0, 1)$ and $(0, -1)$. We check that these two preliminary candidate points can satisfy equation (2). They do, and they do so exactly when $\lambda = 1$, which respects our assumption that $\lambda \neq 2$. So we have candidates:

$f(0, 1) = 1$ at $(0, 1)$ (call it **candidate D**)
$f(0, -1) = 1$ at $(0, -1)$ (call it **candidate E**)

Now look at the five candidates A, B, C, D, and E. Obviously candidate A is the global minimum, and candidates B and C both are global maxima.

EXERCISE 2.
Find the **global** maxima and minima of the function $f(x,y) = xy$ given the constraint $4x^2+y^2 \leq 4$.

A SOLUTION.
First the internal potential extrema, where $4x^2+y^2 < 4$. This only requires finding the critical points, where the function f is horizontal. So compute

$$\text{grad} f = \nabla f = (f_x, f_y) = (y, x).$$

Critical points are where $\nabla f = (y,x) = (0,0)$. So we have the two equations

$$\begin{aligned} &y = 0 \quad \text{and} \\ &x = 0 \end{aligned}$$

So $(0,0)$ is the only critical point and it satisfies $4x^2+y^2 < 4$ and $f(0,0) = 0$. So we have candidate:

$$f(0,0) = 0 \text{ at } (0,0) \quad \text{(call it } \textbf{candidate A}\text{)}$$

Second, the boundary potential extrema, where $g(x,y) = 4x^2+y^2 = 4$. This requires Lagrangian multipliers, where the 'steepest incline' of f on the curve $g = 4$ is parallel to the 'steepest incline' of g. So we get equations

$$\begin{aligned} &\nabla f = \lambda \nabla g \quad \text{and} \\ &g = 4 \end{aligned}$$

More explicitly,

$$\begin{aligned} &(y,x) = \lambda(8x, 2y) \quad \text{and} \\ &4x^2+y^2 = 4 \end{aligned}$$

So solve the three equations in three unknowns:

1. $y = \lambda 8x$
2. $x = \lambda 2y$
3. $4x^2+y^2 = 4$

Equation (1), try $y = 0$. Then equation (3) gives $x = \pm 1$, which works fine with equation (1) by setting $\lambda = 0$. So we may have points $(1,0)$ and $(-1,0)$. We check that these two preliminary candidate points can satisfy equation (2). They do not, because equation (2) will insist that $x = 0$. We must abandon these two points.
Finally, try equation (1) with $y \neq 0$. Then $x \neq 0$, and we can divide by x. So $\lambda = y/8x$. Substitute this for λ in equation (2):

$$x = \frac{2y^2}{8x} = \frac{y^2}{4x}$$

So $4x^2 = y^2$, Then in equation (3) we can replace $4x^2$ by y^2, which gives us $2y^2 = 4$, so $y = \pm\sqrt{2}$. Back to equation $4x^2 = y^2$, we get $4x^2 = 2$, so $x = \pm\sqrt{2}/2$. So we may have points $(\sqrt{2}/2, \sqrt{2})$, $(\sqrt{2}/2, -\sqrt{2})$, $(-\sqrt{2}/2, \sqrt{2})$, and $(-\sqrt{2}/2, -\sqrt{2})$. These points work perfectly fine with equations (1) and (2) by setting $\lambda = 1/4$ when x and y have equal signs, and $\lambda = -1/4$ when x and y have opposite signs. So we have candidates:

$$\begin{aligned} &f(\sqrt{2}/2, \sqrt{2}) = 1 \text{ at } (\sqrt{2}/2, \sqrt{2}) \quad \text{(call it } \textbf{candidate B}\text{)} \\ &f(\sqrt{2}/2, -\sqrt{2}) = -1 \text{ at } (\sqrt{2}/2, -\sqrt{2}) \quad \text{(call it } \textbf{candidate C}\text{)} \\ &f(-\sqrt{2}/2, \sqrt{2}) = -1 \text{ at } (-\sqrt{2}/2, \sqrt{2}) \quad \text{(call it } \textbf{candidate D}\text{)} \\ &f(-\sqrt{2}/2, -\sqrt{2}) = 1 \text{ at } (-\sqrt{2}/2, -\sqrt{2}) \quad \text{(call it } \textbf{candidate E}\text{)} \end{aligned}$$

Now look at the five candidates A, B, C, D, and E. Obviously candidate B and E both are global maxima, and candidates C and D both are global minima.

Exercise 3.

Find the **local** maxima and minima of the function $f(x,y) = xy$ given the constraint $4x^2 + y^2 < 4$.

A solution.

We start with finding the critical points, where the function f is horizontal. So compute

$$\text{grad}f = \nabla f = (f_x, f_y) = (y, x).$$

Critical points are where $\nabla f = (y,x) = (0,0)$. So we have the two equations

$$\begin{aligned} y &= 0 \quad \text{and} \\ x &= 0 \end{aligned}$$

So $(0,0)$ is the only critical point and it satisfies $4x^2 + y^2 < 4$ and $f(0,0) = 0$. So we have just one candidate:

$$f(0,0) = 0 \text{ at } (0,0)$$

Now we have to use the 'second derivative' test, because the domain is the open ellipse without boundary and there may or may not be local extrema. So compute the determinant $D = \det(A) = f_{xx}f_{yy} - f_{xy}f_{yx}$ of the second derivative A of f, which is the matrix

$$A = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Although not too important right now, for all 'nice' functions f it is the case that $f_{xy} = f_{yx}$. Anyway, $f_{xx} = f_{yy} = 0$ and $f_{xy} = f_{yx} = 1$, so $D = -1$. This means that $(0,0)$ is a saddle point, because $D < 0$. Thus f had no local extremum in the given domain $4x^2 + y^2 < 4$.

Exercise 4.
Find the **global** maxima and minima of the function $f(x,y) = 7x + 5y$ given the constraint $x^2 + 4y^2 \leq 8$.

A solution.
First the internal potential extrema, where $x^2 + 4y^2 < 8$. This only requires finding the critical points, where the function f is horizontal. So compute

$$\text{grad}f = \nabla f = (f_x, f_y) = (7,5).$$

Critical points are where $\nabla f = (7,5) = (0,0)$. This is not possible. So there are no internal extrema.

Second, the boundary potential extrema, where $g(x,y) = x^2 + 4y^2 = 8$. This requires Lagrangian multipliers, where the 'steepest incline' of f on the curve $g = 8$ is parallel to the 'steepest incline' of g. So we get equations

$$\nabla f = \lambda \nabla g \quad \text{and}$$
$$g = 8$$

More explicitly,

$$(7,5) = \lambda(2x, 8y) \quad \text{and}$$
$$x^2 + 4y^2 = 8$$

So solve the three equations in three unknowns:

1. $7 = \lambda 2x$
2. $5 = \lambda 8y$
3. $x^2 + 4y^2 = 8$

Equations (1) and (2) immediately imply that x and y and λ all three are not equal to 0. So we can certainly divide by λ. Then equations (1) and (2) give that $2x/7 = 1/\lambda = 8y/5$. So $x = 56y/10$. Plug this in for x in equation (3), and get

$$\frac{56^2 y^2}{100} + 4y^2 = 8$$

or

$$\frac{3536y^2}{100} = 8$$

So $y = \pm\sqrt{\frac{800}{3536}}$. Since $x = 56y/10$ we have matching $x = \pm 56\sqrt{\frac{800}{3536}}/10$.

So we have candidates:

$$f(56\sqrt{\frac{800}{3536}}/10, \sqrt{\frac{800}{3536}}) = (7*56/10+5)\sqrt{\frac{800}{3536}} \text{ at } (56\sqrt{\frac{800}{3536}}/10, \sqrt{\frac{800}{3536}}) \quad \text{(call it } \textbf{candidate A}\text{)}$$

$$f(-56\sqrt{\frac{800}{3536}}/10, -\sqrt{\frac{800}{3536}}) = -(7*56/10+5)\sqrt{\frac{800}{3536}} \text{ at } (-56\sqrt{\frac{800}{3536}}/10, -\sqrt{\frac{800}{3536}}) \quad \text{(call it } \textbf{candidate B}\text{)}$$

Now look at the two candidates A and B. Obviously candidate A is the global maximum, and candidate B is the global minimum. If you are scared of such big expressions, then you may use the calculator and use good decimal approximations.