Write $\sqrt{8 - 2\sqrt{7}} - \sqrt{7}$ as a rational number

Set $c = \sqrt{8 - 2\sqrt{7}} - \sqrt{7}$. We get rid of these nested square roots by rewriting $c + \sqrt{7} = \sqrt{8 - 2\sqrt{7}}$ and squaring both sides:

$$
\begin{aligned}
&(c + \sqrt{7})^2 = 8 - 2\sqrt{7} \\
&c^2 + 2c\sqrt{7} + 7 = 8 - 2\sqrt{7} \\
&c^2 + 2c\sqrt{7} + 2\sqrt{7} - 1 = 0
\end{aligned}
$$

This is a quadratic equation. Rather than going around in circles using the quadratic formula, look for numbers p and q such that

$$c^2 + 2c\sqrt{7} + 2\sqrt{7} - 1 = (c + p)(c + q)$$

So $p + q = 2\sqrt{7}$ and $pq = 2\sqrt{7} - 1$. A little tinkering shows that $p = 1$ and $q = 2\sqrt{7} - 1$ works. So

$$(c + 1)(c + 2\sqrt{7} - 1) = 0$$

So there are exactly two candidates for c to test, namely $c = -1$ and $c = 1 - 2\sqrt{7}$. Optimistically we try $c = -1$ first. We have $-1 =^? \sqrt{8 - 2\sqrt{7}} - \sqrt{7}$ if and only if $\sqrt{7} - 1 =^? \sqrt{8 - 2\sqrt{7}}$. Since both sides are positive, we can square both sides and get the equivalent claim

$$
\begin{aligned}
&(\sqrt{7} - 1)^2 =^? 8 - 2\sqrt{7} \\
&7 - 2\sqrt{7} + 1 =^? 8 - 2\sqrt{7}
\end{aligned}
$$

Yes, these sides are equal. Thus $c = -1$.