MATH 2ddd formalization exercises

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In each of the following exercises you are given a structure and a list of atomic formulas. Then you are given a mathematical statement which you are asked to write in the formal language given by these atomic formulas. In some cases you may be asked to put this formal statement in a special form.

Recall the logical symbols $\wedge$ for and, $\vee$ for or, $\rightarrow$ for implies, $\neg$ for not, $\top$ for true, $\bot$ for false, $\forall$ for for all, and $\exists$ for there exists. Below we always include atom $=$ for equality.

F1. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a constant symbol c standing for an integer, you don't know which. Formalize the statement that c is a multiple of 10.

F2. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a constant symbol c standing for an integer, you don't know which. Formalize the statement that c is not a multiple of 50.

F3. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a constant symbol c standing for an integer, you don't know which. Formalize the statement that c is a multiple of 10 but not a multiple of 50.

F4. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And constant symbols c, d, and e standing for integers, you don't know which ones. Formalize the statement that c is a multiple of d but not a multiple of e.

Another bunch.

F5. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a constant symbol c standing for an integer, you don't know which. Formalize the statement that c is not negative. Write a version that does not use the negation symbol $\neg$.

F6. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And constant symbols c and d standing for integers, you don't know which ones. Formalize the statement that d is positive and c^2 is less than d.

F7. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And constant symbols c and d standing for integers, you don't know which ones. Formalize the statement that d is positive and c is the largest number such that c^2 is less than d.

A bunch over a different structure.

F8. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a function symbol f standing for a function from $\mathbb{Q}$ to $\mathbb{Q}$. Formalize the statement that f is a one-to-one function (injective).

F9. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a function symbol f standing for a function from $\mathbb{Q}$ to $\mathbb{Q}$. Formalize the statement that f is an onto function (surjective).

F10. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a function symbol f standing for a function from $\mathbb{Q}$ to $\mathbb{Q}$. Formalize the statement that f is a one-to-one and onto function (bijective).

F11. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a function symbol f standing for a function from $\mathbb{Q}$ to $\mathbb{Q}$. Formalize the statement that f is a strictly increasing function.

F12. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a function symbol f standing for a function from $\mathbb{Q}$ to $\mathbb{Q}$. Formalize the statement that f is a strictly increasing function which is neither injective nor surjective.

F13. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a function symbol f standing for a function from $\mathbb{Q}$ to $\mathbb{Q}$. Formalize the statement that f is a strictly increasing function which is not bijective.

Yet another bunch.

F14. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. Formalize the statement that all positive numbers are a sum of 4 squares.

F15. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. Formalize the statement that there is a positive number which is not a sum of 3 squares.

F16. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. Formalize the statement that there is a positive number which is a sum of 4 squares but not a sum of 3 squares.

MATH 2ddd proof exercises

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P1. Let $f : A \to B$ and $g : B \to C$ be one-to-one functions. Prove that $g \circ f : A \to C$ is also one-to-one.

P2. Let $f : A \to B$ and $g : B \to C$ be functions such that $g \circ f : A \to C$ is one-to-one. Prove that f is also one-to-one.

P3. Give an example of functions $f : A \to B$ and $g : B \to C$ such that $g \circ f : A \to C$ is one-to-one, but g is not one-to-one.

P4. On finite set $A = \{n \in \mathbb{Z}^+ \ : \ n \leq 100\}$ define binary relation R by $(a, b) \in R$ exactly when a divides b. Prove that R is a partial order (see book, page 302).

P5. On finite set $A = \{n \in \mathbb{Z}^+ \ : \ n \leq 100\}$ define binary relation R by $(a, b) \in R$ exactly when $a - b$ is divisible by 7 in $\mathbb{Z}$. Prove that R is an equivalence relation (see book, page 320 Theorem 3).