MATH 2ddd formalization exercises

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Below starts with a list of problems *with* solutions, as discussed in class. Solve the problems starting from unsolved F12.

Recall the logical symbols $\wedge$ for and, $\vee$ for or, $\rightarrow$ for implies, $\neg$ for not, $\top$ for true, $\bot$ for false, $\forall$ for for all, and $\exists$ for there exists. Below we always include atom $=$ for equality.

F1. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a constant symbol c standing for an integer, you don't know which. Formalize the statement that c is a multiple of 10.

Solution: $\exists x(c = 10 * x)$.

F2. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a constant symbol c standing for an integer, you don't know which. Formalize the statement that c is not a multiple of 50.

Solution: $\neg\exists x(c = 50 * x)$ or $\forall x\, \neg(c = 50 * x)$.

F3. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a constant symbol c standing for an integer, you don't know which. Formalize the statement that c is a multiple of 10 but not a multiple of 50.

Solution: $\exists x(c = 10 * x) \wedge \neg\exists x(c = 50 * x)$.

F4. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And constant symbols c, d, and e standing for integers, you don't know which ones. Formalize the statement that c is a multiple of d but not a multiple of e.

Solution: $\exists x(c = d * x) \wedge \neg\exists x(c = e * x)$.

F5. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a constant symbol c standing for an integer, you don't know which. Formalize the statement that c is not negative. Write a version that does not use the negation symbol $\neg$.

Solution: $c = 0 \ \vee \ 0 < c$.

F6. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And constant symbols c and d standing for integers, you don't know which ones. Formalize the statement that d is positive and c^2 is greater than d.

Solution: $0 < d \ \wedge \ d < c^2$.

F7. We are given the domain of integers $\mathbb{Z}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And constant symbols c and d standing for integers, you don't know which ones. Formalize the statement that d is positive and c is the largest number such that c^2 is less than d.

Solution: $0 < d \ \wedge \ c^2 < d \ \wedge \ \forall x (x^2 < d \rightarrow (x < c \vee x = c))$.

F8. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a function symbol f standing for a function from $\mathbb{Q}$ to $\mathbb{Q}$. Formalize the statement that f is a one-to-one function (injective).

Solution: $\forall x \forall y (f(x) = f(y) \rightarrow x = y)$.

F9. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a function symbol f standing for a function from $\mathbb{Q}$ to $\mathbb{Q}$. Formalize the statement that f is an onto function (surjective).

Solution: $\forall x \exists y (f(y) = x)$.

F10. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a function symbol f standing for a function from $\mathbb{Q}$ to $\mathbb{Q}$. Formalize the statement that f is a one-to-one and onto function (bijective).

Solution: $\forall x \forall y (f(x) = f(y) \rightarrow x = y) \ \wedge \ \forall x \exists y (f(y) = x)$.

F11. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a function symbol f standing for a function from $\mathbb{Q}$ to $\mathbb{Q}$. Formalize the statement that f is a strictly increasing function.

Solution: $\forall x \forall y (x < y \rightarrow f(x) < f(y))$.

F12. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a function symbol f standing for a function from $\mathbb{Q}$ to $\mathbb{Q}$. Formalize the statement that f is (weakly) decreasing.

F13. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. And a function symbol f standing for a function from $\mathbb{Q}$ to $\mathbb{Q}$. Formalize the statement that f has a maximum (nothing is said for which value the maximum is reached).

F14. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. Formalize the statement that all positive numbers are a sum of 4 squares.

F15. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. Formalize the statement that there is a positive number which is not a sum of 3 squares.

F16. We are given the domain of rationals $\mathbb{Q}$, and our language has atoms for equality $=$, for less-than $<$, the usual polynomials with plus, minus, and times, and constants $0, 1, 2, 3, \ldots$. Formalize the statement that there is a positive number which is a sum of 4 squares but not a sum of 3 squares.