Cover page.

H.G. Dales and G. Oliveri (editors). *Truth in Mathematics*, Oxford Science Publications, Clarendon Press — Oxford University Press, Oxford, 1998.

Reviewed by

Wim Ruitenburg

Department of Mathematics
Marquette University
P.O. Box 1881
Milwaukee, WI 53201

This volume includes a record of lectures given at a week-long conference *Truth in Mathematics*, held at Mussomeli, Sicily, Italy, in September of 1995. The contributions are grouped into four parts, following some rough classification. The parts are preceded by a paper by the editors:

IV Sets, undecidability, and the natural numbers

The scholars who contributed chapters to this collection, brought with them significantly varying views on truth in mathematics. Some authors clearly went beyond the subject matter itself. The result is a collection of chapters with significantly varying views on what truth in mathematics is about; a few chapters, although probably of relevance to questions about truth in mathematics, are not about truth in mathematics. The title does not cover all of its contents, but the broad, inclusive, approach resulted in a book that is attractive to read.

Most authors present issues that they and others wrote about elsewhere in greater detail, and hopefully will continue to contribute to in the future. Consequently, one may understand this collection as an overview of some of the approaches to questions about, or related to, truth in mathematics. The individual papers generally give useful further references to the literature. Readers with an interest in foundational or philosophical questions about mathematics will likely find something worthy of their interest. Next, we briefly discuss the individual chapters.

1 **Truth and the foundations of mathematics. An introduction**, *H.G. Dales and G. Oliveri*

This introduction is intended as an attempt to guide the reader to an appreciation of the theme of the book. This is attempted through "a preliminary clarification of the historical background relevant to the contemporary debate on the concept of truth in mathematics; a brief discussion of the mathematical and philosophical importance of such a concept; and a sketch of the applicability of the concept of truth in set theory" [page 1].

The chapter starts with the pre-Tarskian debate, with references to Kant, Frege, Kronecker, Hilbert, and Brouwer. The central section is on Tarski. His work on truth "represents a watershed in the understanding of what it means to say that a statement is true in formalized languages" [page 11]. Next, truth in set theory is discussed. Zermelo's "attempt was the great focus of the debate on what is true in mathematics in the first part of this century" [page 16]. The final subsection discusses the realism versus anti-realism debate.

Although this introduction reflects opinions of the authors, it mostly represents a serious effort to present a balanced view.

2 **Truth and objectivity from a verificationist point of view**, *Dag Prawitz*

This chapter discusses "how truth is to be understood from the point of view of intuitionism or verificationism" [page 41]. Here, meaning is approached from a verificationist point of view rather than on considerations of an ontological kind. The meaning of a statement is determined by what counts as its canonical proof. An indirect proof can then be defined as something that shows that a canonical proof can be given. A statement is true if it is verifiable. The assertion of a proposition requires the actual existence of a proof; the truth of the asserted proposition is identical with the potential existence of a proof.

3 **Constructive truth in practice**, *Douglas S. Bridges*

The essence of constructivism can be expressed by the identification

existence = computability

The rejection of excluded middle is a consequence of this. The author then presents a list of results in constructive analysis which illustrates that constructive mathematics is a viable concern. See also [1].

4 **On founding the theory of algorithms**, *Yiannis N. Moschovakis*

The author has publications on this problem since 1984.

The first sections lay out the problem, in particular the need to precisely define the basic notions of algorithm and implementation. Among the possible approaches to give precise definitions, the author chooses the "standard" one of defining them in set theory. Iterators and recursors are introduced as natural generalizations of the essences of implementation and algorithm. Reduction is introduced as a natural generalization of implementation of algorithm.

The proposed setup suggests interesting extensions to infinitary algorithms.

5 **Truth and knowability: on the principles C and K of Michael Dummett**, *Per Martin-Löf*

When we replace truth by proof, or verification, as the basic notion, we still should try to understand the notion of truth. C and K are principles which, according to Dummett, ought to be satisfied by the notion of the truth of a statement:

C: If a statement is true, there must be something in virtue of which it is true.

K: If a statement is true, it must be in principle possible to know that it is true.

In this paper the author tries to resolve the difficulties inherent in principle K, by clarifying the distinction between the notion of truth of a statement, and the notion of truth of an assertion, or judgment. The author concludes that an assertion is said to be true if it *can* be known, or made evident. A statement is true if it is provable (in the sense of canonical proof). The corrected form of principle K then reads:

> K: If an assertion of the form 'A is true' is correct, then the statement A can be known to be true.

6 **Logical completeness, truth, and proofs**, *Gabriele Lolli*

In practice, mathematicians don't state mathematical truths, but theorems derived from axioms. They establish, in principle, logical truths. The author claims that "[t]he great success of mathematical logic is to have shown that all of logic is independent of a definition of truth (and luckily so, since the latter is undefinable)" [page 119]. By the (classical) completeness theorem of first-order logic, the truth of 'statement A is true' can be interpreted as there exists a proof for A.

The above conclusion does not end debates about computer-assisted proofs or about extremely long finite proofs. Mathematicians could use non-effective methods to show the existence of proofs, for example by resorting to the completeness theorem itself. Although mathematicians give proof sketches, their informal proofs should agree with formal proofs. With reference to the psychology of mathematical thinking, the author claims that "mathematicians see only things or properties describable in words" [page 125].

The author argues that the completeness theorem applies self-referentially to itself, but with some peculiarities. The nature of completeness is more that of an axiom than of a theorem, and axioms are accepted for their usefulness.

7 **Mathematics as language**, *Edward G. Effros*

The author perceives that traditional practices of mathematical proof and rigour are under threat, "due in part to a fundamental misunderstanding of the nature of mathematical thought" [page 131]. The author uses this forum to explain to a wider audience what many mathematicians believe constitutes the 'truth' that may be found in mathematics, and the manner in which many believe that it is threatened by recent developments.

The author gives a brief explanation of why he believes that mathematics is in essence a language, or at least is most valued as a language. For example, the extent to which concepts such as curvature have been adopted by physicists show that the "success of modern physics is in no small part a consequence of the mathematical language [physicists] have at their disposal" [page 132]. They also recognize the significance of a deep understanding of the associated deductive machinery.

The threat to traditional mathematics (in the United States) starts in schools. The author shows evidence in support of his claims that (1) some argue that mathematical fluency is being over-emphasized in schools; (2) proofs are dead in secondary and lower division college education; and (3) some argue that computers have rendered many of the methods of mathematics obsolete.

One effect of these developments is the claim that the basic methods of mathematics, including the method of deduction, are not important to mathematics or to society at large.

The author states that if 'finding the answer' is our only concern, then the purpose of mathematics will be lost. The most valuable product of mathematical research has been discovering concepts. These, in turn, represent extensions in our ability to use language.

8 **Truth, rigour, and common sense**, *Yu. I. Manin*

The author states that "no new insights into [the nature of mathematical truth] have been gained since the epoch of deep discoveries crowned by Gödel's results of the late thirties" [page 147].

Modern mathematics is an essentially linguistic activity. Mathematics is mainly responsible for the structure of deduction. The author puts this claim in a historical perspective: In the final account 'truth' must be a function of the efficiency of social behavior supported by it. Informally, thought allows us to plan behavior (action); subconscious behavior (computation) keeps us alive and kicking. The abstract nature of modern mathematics, as a psychological fact, supports the view of the complementarity of action, thought, and computation.

To the modern working mathematician, (knowing) truth usually means knowing a proof. The practice of mathematics is imprecise, informal. Individually, producing acceptable proofs is hard. Socially, we have to rely upon the work of others. Epistemologically, in principle we can know what a rigorous proof is. In practice we only maintain the current standards of rigor.

The author presents three cases studies. Gödel's proof of the existence of God (1970); the tale of the faulty Pentium computer chip (1994); and Chaitin's claim that a uniform and well defined sequence of mathematical questions can have a 'completely random' sequence of answers (1992 and earlier).

9 **How to be a naturalist about mathematics**, *Penelope Maddy*

The author uses the term naturalism as a metaphilosophical principle describing the proper relations between philosophy and methodology. This chapter is a preliminary sketch of the philosophical aspects of naturalism. See also [2].

Examples of methodological questions are: Is the Continuum Hypothesis CH settled? Or: What proper justification can be offered for a set theory axiom candidate? Variations of realist philosophies claim that there are one or many objective existing worlds of sets. So CH has more or less a truth value. Variations of formalist philosophies, in their 'extreme' form, state that the acceptability CH is not rationally decidable, but comes from æsthetic, psychological, or sociological influences. A fictionalist philosopher may claim that asking CH is like asking how long Hamlet's nose is.

Similar questions, like impredicative definitions and the Axiom of Choice AC, were settled in practice by their 'mathematical fruitfulness,' although the philosophical debates are still going on. Rather than claiming that mathematicians were too hasty in accepting these principles, the author proposes that they have been acceptably evaluated through mathematical practice, rather than through extra-mathematical methods.

Following the lead of historical examples, the naturalist methodologist should exclude what seems methodologically irrelevant, and add a more detailed analysis to the remainder. Extra-mathematical standards are inspirational, not justificatory. This should bring out the implicit means/end considerations, with the aim of drawing out sound methodological arguments. When applied to the case of CH, a naturalist may conclude that there is no mathematically sound basis on which to settle CH.

The words of a practitioner should not be taken as gospel, but the only likely possibility of error occurs when "extra-mathematical considerations [...] confuse and distort methodological discussions" [page 171]. Naturalism has purely descriptive aspects and sociology of science aspects at its base, but it continues by applying mathematical practice itself at its next stage. Finally, history and mathematical practice provide a further analysis.

For a naturalistic philosopher there is still room for questions like what is the relationship between mathematics and science? Methodologically, there is not. Given that constraint, certain subtle versions of platonism, formalism, or fictionalism, may be acceptable to a naturalist philosopher as scientific theories of mathematical practice. Mathematics differs from astrology in that mathematics offers very useful models for spatiotemporal events, but does not overlap the domain of science.

10 **The mathematician as a formalist**, *H.G. Dales*

The author claims that "there is no reason to suppose that mathematicians have an innate understanding of the philosophical foundations of their subject, or even any coherent and well-thought-out view of what exactly they are engaged in" [page 181]. Philosophers of mathematics must take into account the collection of theorems that mathematicians have proved, as well as the style of presenting mathematics that is the current orthodoxy. The author intends to describe how mathematicians may act as formalists when they are writing down their mathematics.

Comparing realism and formalism, it appears that, informally speaking, most mathematicians are realists on weekdays, and formalists on Sundays. However, this realism may only be psychological, for the proofs are essentially formal. 20^{th} century mathematics, since about 1930, has been formalist. Mathematicians believe in the formalizability-in-principle of their results.

The formalist has the freedom to choose any axiom system as long as it is consistent. This still leaves the question of why certain choices are preferred. In his answer the author follows the naturalist methodological approach. Additionally, the author gives four criteria that formalists should adopt.

Criterion I: Axioms should be simple and clear, and should isolate the essential aspects of many diverse, known examples; the choice will have been successful if they are fecund in suggesting other, new examples, and in encompassing examples which arise in other contexts.

Criterion II: The depth of the development that takes place within the subject specified by the axioms.

Criterion III: The frankly æsthetic one.

Criterion IV: One should not arbitrarily restrict the notions under consideration unless forced to do so by the desire to avoid contradiction.

11 **A credo of sorts**, *V.F.R. Jones*

The author demonstrates how 'ordinary' mathematicians can live with worries like Russell's paradox. Through the example of the Fourier transform, the author shows how a physicist knows that the transform is 'correct' from the tangible evidence of his science. In some sense theoretical physicists are to mathematicians, as mathematicians are to logicians. If an inconsistency does show up, the 'essential correctness' is expected to survive. Progress in mathematics will never follow any rules imposed upon it. When a difficult theorem is proven, its author is usually partly convinced by the circumstantial evidence. Although proofs are indispensable, it is fair to say they are necessary but not sufficient for mathematical truth, at least truth as perceived by the individual. Additionally, the test of time is part of the acceptance process of such theorems.

12 **Mathematical evidence**, *Donald A. Martin*

The author gives two examples from descriptive set theory of how one may decide on what is evidence for mathematical truth. Both are cases of strong scientific evidence for the truth of propositions. In this chapter, the preferred frame of mind is one that considers truth and evidence in a direct and unanalyzed way.

Mathematical evidence comes from proof. In current mathematics, to rigorously prove a statement, the obvious interpretation is that one must show that it follows by pure logic (first-order logic) from the basic principles of mathematics (the ZFC axioms of set theory). Although the author is skeptical that the truth of each of the axioms of ZFC is known with certainty, ZFC appears to fairly reflect the axiomatization of the iterative concept of set.

Although adequate for most of mathematics, ZFC turns out to be inadequate (seriously incomplete) for set theory itself. Proof is an obvious example of what is 'proper mathematical' evidence that counts towards giving mathematical knowledge of the truth of a proposition. Probabilistic arguments may not qualify as such. Proper mathematical evidence for the two examples below include (1) support for (new) fundamental axioms; and (2) a richness of evidence analogous to evidence for general theorical statements in empirical sciences.

Whatever proper mathematical evidence there may be in support of the Axiom of Determinacy AD, it contradicts the well-entrenched Axiom of Choice AC. AD, even when restricted to special sets, implies the (Turing) Cone Lemma and Wadge's Lemma for these special sets. These examples show that AD, whether restricted to special sets or not, (1) presents abundant consequences; (2) sheds light on a discipline; and (3) offers powerful methods for solving problems.

One may object to support for (restricted) AD because it is extrinsic (it is not motivated by the iterative concept of set). But certain (very) large cardinal hypotheses imply the restricted versions of AD, so these should then also be interpreted as extrinsic. Moreover, intrinsic evidence does not explain the properness of the evidence for the standard ZFC axioms.

13 **Mathematical definability**, *Theodore A. Slaman*

The author gives an introduction to the hierarchy of definability in first- and second-order arithmetic. While ignoring distinctions of finite size, a short treatment is given of those aspects of definability which are tied to axiomatic set theory and the large cardinal hierarchy.

Applications illustrate the utility of a detailed structure theory for definability. Insights into definability lead to insights into provability within second-order arithmetic.

The author claims that there is strong evidence that this is the only hierarchy of definability.

14 **True to the pattern**, *Gianluigi Oliveri*

The author sharpens and defends the view of Wittgenstein that an aspect (or pattern) is not a property of the object, but an internal relation between it and other objects. This conception can be used to establish that aspects are real. and it shows that if mathematics is a science of patterns, the conception of truth best fitting with it is that of Aristotle and Tarski.

The author considers the applicability of the traditional Kantian notions of seeing and interpreting as unsatisfactory. Rather, 'aspect seeing' is a characteristic of the imagination that imposes structure over sensory input by means of concepts. The concept of 'internal relation' gives a more satisfactory characterization of mathematical experience. Here a relational property Φ is internal to a term A only when A has property Φ and it is necessary that for any x, if x does not have property Φ, then x is different from A.

The author's model of experience has the following characteristics: Although perceptual and intellectual faculties of reason remain distinct, some of the theoretical vehicles through which such faculties are exercised (concepts) are shared by them. Such shared theoretical vehicles are not a priori in the mind, but are the outcome of the cultural activity of human kind. Patterns are real, but they are not properties of objects.

15 **Foundations of set theory**, *W.W. Tait*

According to the iterative conception of set theory, sets are the objects in any member of the hierarchy of domains obtained from the null domain by iterating the power set operation.

Gödel stated that new axioms for set theory can be accepted based on their success in the sense of fruitfulness in consequences, in particular in consequences demonstrable without the new axiom; whose proofs with the help of the new axiom are considerably simpler and easier to discover; and make it possible to condense into one proof many different proofs. Although such attempts may result in interesting new theories, their success would not necessarily be justified by the iterative concept. So these new theories are not about (iterative) set theory.

Over a set theory language with notation for class-level relations, the author introduces levels $\mathrm{RF}(n, m, k)$ of reflection principles of the form

$$\forall X, \ldots, Y[\varphi(X, \ldots, Y) \rightarrow \exists\beta\varphi^{\beta}(X^{\beta}, \ldots, Y^{\beta})],$$

where n, m, and k are bounds on the complexity of φ. An $\mathrm{RF}(1, 0, 2)$ reflection principle implies the (second-order) Axiom of Replacement.

Although classes are often treated like sets, the universe of sets as a completed totality is rejected. So the notion of truth over the class of all sets requires explanation. The author argues that the notion of truth for this and other classes should not be regarded as determined for every sentence. The (formal) logic that applies should be constructive (intuitionistic) logic and type theory. As a consequence, the reflection principle must be restricted to ranges over decidable objects.

The author shows the strengths of certain subcollections of the reflection principle.

16 **Which undecidable mathematical sentences have determinate truth values?**, *Hartry Field*

Plenitudinous platonism essentially says that each consistent mathematical theory has a model under the standard satisfaction relation. Methodologically, plenitudinous platonism is anti-objectivistic in that, for example, sets that satisfy conceptions other than 'our own,' do not require an explanation in terms of 'our' sets. Methodologically, plenitudinous platonism is like fictionalism.

Assuming we intend to have some full universe in mind, our 'fullest' mathematical theory may still be incomplete. This raises the following objectivity issue [page 294]:

> Which undecidable mathematical sentences have determinate truth values?

Our logic has finiteness properties (formulas and proofs are finite objects), so at least we wish determinate truth values for finitude. Set theory is questionable, but 'determinacy of finitude' would help with arithmetic to get determinacy beyond Peano Arithmetic.

The author claims that it is sufficient to assume the cosmological principles of time being (A) infinite in extent and (B) Archimedean, to extend determinacy in the physical vocabulary to the notion of finiteness.

The concept of 'fullest' mathematical theory has to be somewhat vague. In extreme anti-objectivism, it may be indeterminate whether our fullest theory is consistent. The usual premises of the inductive argument for the truth of all theorems cannot all be accepted.

17 **Two conceptions of natural number**, *Alexander George and Daniel J. Velleman*

The authors consider two introductions of natural number, the build up (BU) version (start with 0 and keep closing under successor) and the pair down (PD) version (the intersection of all sets containing 0 and closed under successor). PD is problematic from the constructivist point of view. BU is problematic from the platonist point of view (the finiteness of iteration is not satisfactorily captured by the definition).

Following Charles Parsons, induction should be understood as referring to arbitrary predicates. This includes predicates as yet undefined.

Both definitions BU and PD presuppose some understanding of the very concepts being defined.

18 **The tower of Hanoi**, *W. Hugh Woodin*

Let $\exp(1,n) = n$, and $\exp(k+1,n) = 2^{\exp(k,n)}$. There is a sentence $\Omega(x)$ for set theory which implicitly defines a property for finite sequences, such that:

> For a given sequence this property is easily decided.
> For each sequence s of length n with this property, its elements consist of non-negative integers less than n, and its verification can be completed in fewer than n^2 steps.
> If there exists a sequence of length n with this property, then $\exp(2011,n)$ does not exist.

There are models where such (non-standard) n exist. Assuming that arbitrarily large sets can exist there is, for each suitable n, no proof of length less than n that no such sequence of length n can have this property.

There are limitations to the extent our experience in mathematics to date refutes the existence of such sequences. The author argues that a consistent philosophical view must acknowledge the possibility that such sequences of length $n = 10^{24}$ or so could exist, just as those who study large cardinals must admit the possibility that the notions are not consistent.

The author introduces a weak theory T_0 and sentences Ω_k such that Ω_k essentially asserts that there is a proof from T_0 of $\neg\Omega_k$ of length less than $n = 10^{24k}$. In our universe of sets $\neg\Omega_k$ is true.

References

[1] Bishop, Errett; Douglas Bridges. *Constructive Analysis*. Grundlehren der mathematischen Wissenschaften **279**, Springer-Verlag, 1985.

[2] Maddy, P. *Naturalism in Mathematics*. Clarendon Press, Oxford, 1997.