Kripke Submodels and Universal Sentences

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Abstract

We define two notions for intuitionistic predicate logic: that of a submodel of a Kripke model, and that of a universal sentence. We then prove a corresponding preservation theorem. If a Kripke model is viewed as a functor from a small category to the category of all classical models with (homo)morphisms between them, then we define a submodel of a Kripke model to be a restriction of the original Kripke model to a subcategory of its domain where every node in the subcategory is mapped to a classical submodel of the corresponding classical model in the range of the original Kripke model. We call a sentence universal if it is built inductively from atoms (including $\top$ and $\bot$) using $\wedge$, $\vee$, $\forall$, and $\rightarrow$, with the restriction that antecedents of $\rightarrow$ must be atomic. We prove that an intuitionistic theory is axiomatized by universal sentences if and only if it is preserved under Kripke submodels. We also prove the following analogue of a classical model-consistency theorem: The universal fragment of a theory Γ is contained in the universal fragment of a theory Δ if and only if every rooted Kripke model of Δ is strongly equivalent to a submodel of a rooted Kripke model of Γ. Our notions of Kripke

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submodel and universal sentence are natural in the sense that in the presence of the rule of excluded middle, they collapse to the classical notions of submodel and universal sentence.

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1 Introduction

In his paper, "Submodels of Kripke Models" [6], Albert Visser suggests several different notions of a submodel of a Kripke model for intuitionistic predicate logic: First, if a Kripke model is viewed as a functor from an arbitrary small category to the category of all classical models with morphisms between them, then one might consider a submodel to be a functor defined on the same domain (category) of the original Kripke model, but where every node is mapped to a classical submodel of the corresponding classical model in the range of the original Kripke model. Second, one might consider a submodel to be the restriction of the original Kripke model (as a functor) to a full subcategory of its domain. Third, one might consider a submodel to be a combination of the first and second notions. In [6], Visser chooses the second notion (above), and proves that an intuitionistic theory is preserved under Kripke submodels if and only if it is axiomatized by semipositive sentences. The first notion is considered by Bagheri and Moniri in [1].

When defining notions of Kripke submodel and universal sentence for intuitionistic logic, it seems reasonable to demand the following properties: First, both the notion of submodel and universal sentence should include the classical notions as special cases and should reduce to them in the case where the law of excluded middle is included as an axiom schema in the base theory, in which case intuitionistic predicate logic and classical predicate logic coincide. Second, an intuitionistic theory $\Delta \supseteq \Gamma$ should be preserved under Kripke submodels that satisfy a base theory Γ if and only if Δ is axiomatizable by universal sentences over Γ. Third, given theories Γ and Δ, it should be the case that the universal fragment of Γ is contained in the universal fragment of Δ if and only if every model of Δ is contained in a model of Γ, in which case Γ is said to be model-consistent relative to Δ, see [3].

To this end, we define a submodel of a Kripke model to be a restriction of the original Kripke model (viewed as a functor) to a subcategory of its domain, but where every node in the subcategory is mapped to a classical submodel of the corresponding classical model in the range of the original Kripke model. This notion corresponds to the third suggestion of Visser's, and it includes the first and second notions as special cases. In the case where the law of excluded middle is axiomatized in the base theory, the Kripke models involved become essentially classical models, in that forcing at a node coincides with satisfaction in the corresponding classical model, and all morphisms are elementary embeddings. If such Kripke models are rooted, it follows that the sentences true in the Kripke model are precisely the sentences classically true at the root, and in this case our notion of submodel coincides with the classical notion. We also define a class of universal sentences for intuitionistic predicate logic that coincides (up to provable equivalence) with the class of Π^0_1 sentences of classical logic in the presence of the law of excluded middle. We call a sentence universal if it is built inductively from atomic sentences (including $\top$ and $\bot$) using $\wedge$, $\vee$, $\forall$, and $\rightarrow$, with the restriction that antecedents of $\rightarrow$

must be atomic. We prove that a theory is axiomatized by universal sentences if and only if it is preserved under Kripke submodels. Our proof is based on Visser's proof in [6]. We also prove an intuitionistic analogue of a classical model-consistency theorem, using our notions of universal sentence and Kripke submodel: Given intuitionistic theories Γ and Δ, the universal fragment of Γ is contained in the universal fragment of Δ if and only if every rooted Kripke model of Δ is strongly equivalent to a submodel of a rooted Kripke model of Γ. The notion of strong equivalence is defined in Section 4.

We consider a first order language $\mathcal{L}$ to be the set of formulas that can be built from a symbol set (variables, relation, function, and constant symbols) using $\top$, $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\exists$, and $\forall$. In the present paper, we consider only languages that include $=$ as a binary relation, interpreted as real equality in a model. Symbols $\top$ and $\bot$ are both atoms and nullary connectives. Negation $\neg\varphi$ is short for $\varphi \rightarrow \bot$, and bi-implication $\varphi \leftrightarrow \psi$ is short for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. We use the bold font ($\mathbf{A}$) to denote categories, the fraktur font ($\mathfrak{A}$) to denote first order classical models and Kripke models, and the calligraphy font ($\mathcal{A}$) to denote languages and subsets of languages. In addition, we make use of the following notational conventions: A list of constant symbols or variables $t_1, ..., t_n$ is abbreviated as $\mathbf{t}$. If C is an arbitrary set of constants, then $\mathcal{L}(C)$ is the language $\mathcal{L}$ extended by all constants in C. $\mathcal{A}\text{t} \subseteq \mathcal{L}$ is the set of atomic formulas in $\mathcal{L}$. Analogously, $\mathcal{A}\text{t}(C) \subseteq \mathcal{L}(C)$ is the set of atomic formulas in $\mathcal{L}(C)$, and so on. Given a classical model $\mathfrak{A}$, the domain of $\mathfrak{A}$ is denoted A, and $\mathcal{L}(A)$ is the language $\mathcal{L}$ extended by a new constant for every element in A. The symbol $\models$ denotes classical satisfaction in a model, and it is defined for sentences (closed formulas) only. $\text{Th}(\mathfrak{A}) = \{\varphi \in \mathcal{L}(A) : \mathfrak{A} \models \varphi\}$ is the elementary diagram of $\mathfrak{A}$. This notation is convenient, since we can write $\text{Th}(\mathfrak{A}) \cap \mathcal{L}$ for the complete theory of $\mathfrak{A}$ over $\mathcal{L}$, $\text{Th}(\mathfrak{A}) \cap \mathcal{A}\text{t}(A)$ for the positive atomic diagram of $\mathfrak{A}$, and so on. The symbol $\vdash$ denotes intuitionistic derivability, and is defined for sentences only. If $\Gamma \subseteq \mathcal{L}$ is a set of sentences, then $\text{Th}(\Gamma) = \{\varphi \in \mathcal{L} : \Gamma \vdash \varphi\}$ is the deductive closure of Γ over $\mathcal{L}$. If C is an arbitrary set of constants and $\Gamma \subseteq \mathcal{L}(C)$ is a set of sentences, then $\text{Th}[C](\Gamma) = \{\varphi \in \mathcal{L}(C) : \Gamma \vdash \varphi\}$ is the deductive closure of Γ over $\mathcal{L}(C)$. We consider a theory to be a set of sentences closed under (intuitionistic) deduction.

2 Kripke Models

Let $\mathcal{L}$ be a first order language, and let $\mathbf{M}(\mathcal{L})$ be the category of all classical models for the language $\mathcal{L}$, with all (homo)morphisms between them. That is, a morphism in this category is a classical homomorphism in the sense of [4] and [2]. Let $\mathbf{A}$ be an arbitrary small category (in practice, $\mathbf{A}$ is often taken to be a small poset category). A Kripke model $\mathfrak{A}$ is a functor $\mathfrak{A} : \mathbf{A} \rightarrow \mathbf{M}(\mathcal{L})$. So for every object $i \in |\mathbf{A}|$, there is an associated classical model $\mathfrak{A}(i) = \mathfrak{A}_i$ in $\mathbf{M}(\mathcal{L})$, and for every arrow $f : i \rightarrow j$ in $\mathbf{A}$, there is an associated morphism $\mathfrak{A}(f)$ $= \mathfrak{A}f : \mathfrak{A}_i \rightarrow \mathfrak{A}_j$. Contrary to [5], we interpret the equality predicate as real equality in each node structure $\mathfrak{A}_i$. The existence of a morphism $\mathfrak{A}f : \mathfrak{A}_i \rightarrow \mathfrak{A}_j$ means essentially that $\mathfrak{A}_j$ models the positive atomic diagram of $\mathfrak{A}_i$. That is, for all sentences $\varphi(\mathbf{a}) \in \mathcal{A}\text{t}(A_i)$ with $\mathbf{a} \in A_i$, if $\mathfrak{A}_i \models \varphi(\mathbf{a})$ then $\mathfrak{A}_j \models \varphi(\mathfrak{A}f(\mathbf{a}))$.

Let $\mathfrak{A} : \mathbf{A} \rightarrow \mathbf{M}(\mathcal{L})$ be a Kripke model. For every node $i \in |\mathbf{A}|$ and for every sentence $\varphi \in \mathcal{L}(A_i)$, we define the forcing relation $\Vdash^{\mathfrak{A}}$ inductively by:

$$i \Vdash^{\mathfrak{A}} \varphi \;\Leftrightarrow\; \mathfrak{A}_i \models \varphi, \quad \text{for all (atomic) sentences } \varphi \in \mathcal{A}\text{t}(A_i),$$
$$i \Vdash^{\mathfrak{A}} \varphi \wedge \psi \;\Leftrightarrow\; i \Vdash^{\mathfrak{A}} \varphi \text{ and } i \Vdash^{\mathfrak{A}} \psi,$$

$i \Vdash^{\mathfrak{A}} \varphi \vee \psi \;\Leftrightarrow\; i \Vdash^{\mathfrak{A}} \varphi$ or $i \Vdash^{\mathfrak{A}} \psi$,
$i \Vdash^{\mathfrak{A}} \varphi \to \psi \;\Leftrightarrow\;$ for all $f : i \to j$, if $j \Vdash^{\mathfrak{A}} \varphi^f$ then $j \Vdash^{\mathfrak{A}} \psi^f$,
$i \Vdash^{\mathfrak{A}} \forall x \varphi(x) \;\Leftrightarrow\;$ for all $f : i \to j$ and for all $a \in A_j$, $j \Vdash^{\mathfrak{A}} \varphi^f(a)$,
and
$i \Vdash^{\mathfrak{A}} \exists x \varphi(x) \;\Leftrightarrow\; i \Vdash^{\mathfrak{A}} \varphi(a)$ for some $a \in A_i$,

where $\varphi^f \in \mathcal{L}(A_j)$ is constructed from $\varphi \in \mathcal{L}(A_i)$ by replacing all constant symbols $a \in A_i$ in φ by $\mathfrak{A}f(a) \in A_j$.

We say that a sentence $\varphi \in \mathcal{L}(A_i)$ is *true* at node $i \in |\mathbf{A}|$ if $\mathfrak{A}_i \models \varphi$. We say that a sentence $\varphi \in \mathcal{L}(A_i)$ is *forced* at node $i \in |\mathbf{A}|$ if $i \Vdash^{\mathfrak{A}} \varphi$. We say that a sentence $\varphi \in \mathcal{L}$ is forced in the Kripke model $\mathfrak{A}$, written $\mathfrak{A} \Vdash \varphi$, if $i \Vdash^{\mathfrak{A}} \varphi$ for all $i \in |\mathbf{A}|$. If $\Gamma \subseteq \mathcal{L}$ is a set of sentences, then $\mathfrak{A} \Vdash \Gamma$ if and only if $\mathfrak{A} \Vdash \varphi$ for all $\varphi \in \Gamma$. The Kripke model $\mathfrak{A}$ is *rooted* if there exists an $i_0 \in |\mathbf{A}|$ such that for all $i \in |\mathbf{A}|$, there is an $f : i_0 \to i$ in $\mathbf{A}$. It is easy to verify that sentences in Kripke models are *persistent*: that is, for all $f : i \to j$ in $\mathbf{A}$ and for all $\varphi \in \mathcal{L}(A_i)$, if $i \Vdash^{\mathfrak{A}} \varphi$ then $j \Vdash^{\mathfrak{A}} \varphi^f$. In the case of a rooted Kripke model $\mathfrak{A}$ with root i_0, we have $\mathfrak{A} \Vdash \varphi$ if and only if $i_0 \Vdash^{\mathfrak{A}} \varphi$.

Let C be an arbitrary set of constants. A theory Γ over $\mathcal{L}(C)$ is called *prime* if for all sentences $\varphi, \psi \in \mathcal{L}(C)$, we have $\Gamma \vdash \varphi \vee \psi$ if and only if $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$. A consistent theory Γ over $\mathcal{L}(C)$ is called *C-Henkin* if for all sentences $\exists x \varphi(x) \in \mathcal{L}(C)$, we have $\Gamma \vdash \exists x \varphi(x)$ if and only if there is a $c \in C$ such that $\Gamma \vdash \varphi(c)$. A theory is called *C-Henkin prime* if it is both C-Henkin and prime.

The next two results are basic to intuitionistic logic:

Proposition 2.1 *Let $\mathcal{L}$ be a first order language, let C be a set of constants not in $\mathcal{L}$, with $|C| \geq |\mathcal{L}|$, let $\varphi \in \mathcal{L}$ be a sentence, and let Γ be a theory over $\mathcal{L}$ such that $\Gamma \nvdash \varphi$. Then there is a C-Henkin prime theory Γ' over $\mathcal{L}(C)$ such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash \varphi$.*

Proof. See [2], Section 5.3. $\dashv$

Proposition 2.2 *Let $\mathcal{L}$ be a first order language, let C be a set of constants, and let Γ be a C-Henkin prime theory over $\mathcal{L}(C)$. Then there is a rooted Kripke model $\mathfrak{A}$ over $\mathcal{L}(C)$ such that*

$$\mathfrak{A} \Vdash \varphi \Leftrightarrow \Gamma \vdash \varphi, \quad \textit{for all } \varphi \in \mathcal{L}(C).$$

Proof. See [2], Section 5.3. $\dashv$

3 Kripke Submodels and Universal Sentences

If $\mathfrak{A} : \mathbf{A} \to \mathbf{M}(\mathcal{L})$ and $\mathfrak{B} : \mathbf{B} \to \mathbf{M}(\mathcal{L})$ are Kripke models, then $\mathfrak{A}$ is a *submodel* of $\mathfrak{B}$ if and only if $\mathbf{A}$ is a subcategory of $\mathbf{B}$, and for all $i \in |\mathbf{A}|$, the structure $\mathfrak{A}_i$ is a classical submodel of $\mathfrak{B}_i$. That is, $\mathfrak{A} \subseteq \mathfrak{B}$ if and only if $\mathbf{A} \subseteq \mathbf{B}$ and $\mathfrak{A}_i \subseteq \mathfrak{B}_i$ for all $i \in |\mathbf{A}|$. We also say that $\mathfrak{B}$ is an *extension* of $\mathfrak{A}$.

Let $\mathcal{L}$ be a first order language. We define the set of universal formulas $\mathcal{U} \subseteq \mathcal{L}$ inductively by:

$\varphi \in \mathcal{A}\mathrm{t} \;\Rightarrow\; \varphi \in \mathcal{U}$,
$\varphi, \psi \in \mathcal{U} \;\Rightarrow\; \varphi \wedge \psi,\ \varphi \vee \psi \in \mathcal{U}$,
$\varphi \in \mathcal{A}\mathrm{t},\ \psi \in \mathcal{U} \;\Rightarrow\; \varphi \to \psi \in \mathcal{U}$, and
$\varphi \in \mathcal{U} \;\Rightarrow\; \forall x \varphi \in \mathcal{U}$.

The set of positive existential formulas $\mathcal{E}^+ \subseteq \mathcal{L}$ is the set containing $\mathcal{A}$t and closed under $\wedge$, $\vee$, and $\exists$. Following Troelstra and van Dalen [5], we denote intuitionistic predicate logic by IQC. We note the following:

$$\begin{aligned}&\mathrm{IQC} \vdash ((\varphi \wedge \psi) \to \theta) \leftrightarrow (\varphi \to (\psi \to \theta)),\\ &\mathrm{IQC} \vdash ((\varphi \vee \psi) \to \theta) \leftrightarrow ((\varphi \to \theta) \wedge (\psi \to \theta)), \quad \text{and}\\ &\mathrm{IQC} \vdash (\exists x \varphi(x) \to \psi) \leftrightarrow \forall x(\varphi(x) \to \psi)\end{aligned}$$

(where in the third case, x is not free in ψ). Using these intuitionistic tautologies, it is easy to see that any sentence of the form $\varphi \to \psi$, where $\varphi \in \mathcal{E}^+$ and $\psi \in \mathcal{U}$, is provably equivalent (over IQC) to a sentence in $\mathcal{U}$. Also, in the presence of the law of excluded middle, the set $\mathcal{U}$ becomes the set of classical Π^0_1 formulas.

First we prove the following simple result, which is one direction of our preservation theorem.

Theorem 3.1 *Let $\Gamma \subseteq \Delta$ be intuitionistic theories over a language $\mathcal{L}$, and suppose that Δ is axiomatizable by universal sentences over Γ. Then for all Kripke models $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$, if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \Vdash \Delta$. That is, Δ is preserved under Γ-Kripke submodels.*

Proof. Suppose that Δ is axiomatized by universal sentences over Γ. Let $\mathfrak{A} : \mathbf{A} \to \mathbf{M}(\mathcal{L})$ and $\mathfrak{B} : \mathbf{B} \to \mathbf{M}(\mathcal{L})$ be Kripke models such that $\mathfrak{A} \Vdash \Gamma$, $\mathfrak{B} \Vdash \Delta$, and $\mathfrak{A} \subseteq \mathfrak{B}$. It suffices to show that $\mathfrak{A} \Vdash \Delta \cap \mathcal{U}$. This will follow if we show that for all $i \in |\mathbf{A}|$ and for all sentences $\varphi \in \mathcal{U}(A_i)$, if $i \Vdash^{\mathfrak{B}} \varphi$ then $i \Vdash^{\mathfrak{A}} \varphi$. The proof is by induction on the complexity of φ, for all i simultaneously. Let $i \in |\mathbf{A}|$, and let $\varphi \in \mathcal{U}(A_i)$ be a sentence. Suppose $\varphi \in \mathcal{A}\mathrm{t}(A_i)$ is atomic, and $i \Vdash^{\mathfrak{B}} \varphi$. Since $A_i \subseteq B_i$, we have $\varphi \in \mathcal{A}\mathrm{t}(A_i) \subseteq \mathcal{A}\mathrm{t}(B_i)$. So $\mathfrak{B}_i \models \varphi$. Thus, since φ is quantifier free, and $\mathfrak{A}_i \subseteq \mathfrak{B}_i$, we have $\mathfrak{A}_i \models \varphi$. So $i \Vdash^{\mathfrak{A}} \varphi$. The induction steps for $\varphi := \psi \wedge \theta$ and $\varphi := \psi \vee \theta$ are obvious. Suppose $i \Vdash^{\mathfrak{B}} \psi \to \theta$, where $\psi \in \mathcal{A}\mathrm{t}(A_i)$. Let $g : i \to j$ be in $\mathbf{A}$. Suppose $j \Vdash^{\mathfrak{A}} \psi^g$. Since $\psi^g \in \mathcal{A}\mathrm{t}(A_j)$, we have $\mathfrak{A}_j \models \psi^g$. Since $\mathfrak{A}_j \subseteq \mathfrak{B}_j$, we have $\mathfrak{B}_j \models \psi^g$. Since $A_j \subseteq B_j$, we have $\psi^g \in \mathcal{A}\mathrm{t}(A_j) \subseteq \mathcal{A}\mathrm{t}(B_j)$. So $j \Vdash^{\mathfrak{B}} \psi^g$. Since $\mathbf{A} \subseteq \mathbf{B}$, $g : i \to j$ is in $\mathbf{B}$. Thus, since $i \Vdash^{\mathfrak{B}} \psi \to \theta$ and $j \Vdash^{\mathfrak{B}} \psi^g$, we have $j \Vdash^{\mathfrak{B}} \theta^g$. So by induction hypothesis, $j \Vdash^{\mathfrak{A}} \theta^g$. So for all $g : i \to j$ in $\mathbf{A}$, if $j \Vdash^{\mathfrak{A}} \psi^g$ then $j \Vdash^{\mathfrak{A}} \theta^g$. Thus, $i \Vdash^{\mathfrak{A}} \psi \to \theta$. Now suppose $i \Vdash^{\mathfrak{B}} \forall x \varphi(x)$. Let $g : i \to j$ be in $\mathbf{A}$. Let $a \in A_j \subseteq B_j$. Since $\mathbf{A} \subseteq \mathbf{B}$, $g : i \to j$ is in $\mathbf{B}$. So $j \Vdash^{\mathfrak{B}} \varphi^g(a)$. By induction hypothesis, we have $j \Vdash^{\mathfrak{A}} \varphi^g(a)$. So for all $g : i \to j$ in $\mathbf{A}$ and for all $a \in A_j$, we have $j \Vdash^{\mathfrak{A}} \varphi^g(a)$. Thus, $i \Vdash^{\mathfrak{A}} \forall x \varphi(x)$. This completes the induction on the complexity of φ. $\dashv$

Next we prove a lemma that is used heavily in the subsequent result. First we give a definition:

Definition 3.2 *Let $\mathcal{L}$ be a first order language, let C and D be sets of constants with $C \subseteq D$, let Γ be a consistent theory over $\mathcal{L}(C)$ and let Δ be a consistent theory over $\mathcal{L}(D)$. The quadruple $\langle \Gamma, C, D, \Delta \rangle$ is called* **acceptable** *if $\Gamma \cap \mathcal{A}\mathrm{t}(C) \subseteq \Delta$ and $\Delta \cap \mathcal{U}(C) \subseteq \Gamma$.*

Lemma 3.3 *Let $\mathcal{L}$ be a first order language, let C and D be sets of constants with $C \subseteq D$, let Γ be a consistent theory over $\mathcal{L}(C)$ and let Δ be a consistent theory over $\mathcal{L}(D)$. If $\Delta \cap \mathcal{U}(C) \subseteq \Gamma$, then the quadruple $\langle \Gamma, C, D, \mathrm{Th}[D](\Delta \cup (\Gamma \cap \mathcal{A}\mathrm{t}(C))) \rangle$ is acceptable.*

Proof. Let $\Delta' = \mathrm{Th}[D](\Delta \cup (\Gamma \cap \mathcal{A}\mathrm{t}(C)))$. Obviously, $\Gamma \cap \mathcal{A}\mathrm{t}(C) \subseteq \Delta'$. We must show that $\Delta' \cap \mathcal{U}(C) \subseteq \Gamma$. Let $\varphi \in \Delta' \cap \mathcal{U}(C)$. Then $\Delta \cup (\Gamma \cap \mathcal{A}\mathrm{t}(C)) \vdash \varphi$. By compactness, we have $\Delta \cup \{\rho\} \vdash \varphi$, where ρ is a conjunction of atoms in $\Gamma \cap \mathcal{A}\mathrm{t}(C)$. So $\Delta \vdash \rho \to \varphi$. Since $\rho \in \mathcal{E}^+(C)$ and $\varphi \in \mathcal{U}(C)$, we have that IQC $\vdash (\rho \to \varphi) \leftrightarrow \psi$ for some $\psi \in \mathcal{U}(C)$. Thus, $\psi \in \Delta \cap \mathcal{U}(C) \subseteq \Gamma$. So $\Gamma \vdash \rho \to \varphi$. Also $\Gamma \vdash \rho$. So $\Gamma \vdash \varphi$. Since $\bot \in \mathcal{U}$ and Γ is consistent, Δ' is also consistent. ⊣

The next two results are basic to the results of the next section:

Proposition 3.4 *Let $\mathcal{L}$ be a first order language, let C and D be sets of constants, and let Γ and Δ be theories such that $\langle \Gamma, C, D, \Delta \rangle$ is acceptable. Let E be a set of constants not in $\mathcal{L}(D)$, with $|E| \geq |\mathcal{L}(D)|$, and let $\varphi \in \mathcal{L}(C)$ be a sentence such that $\Gamma \nvdash \varphi$. Then there is an acceptable quadruple $\langle \Gamma', C', D', \Delta' \rangle$ such that Γ' is C'-Henkin prime, Δ' is D'-Henkin prime, $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, $C \subseteq C'$, $D \subseteq D' \subseteq D \cup E$, and $\Gamma' \nvdash \varphi$.*

Proof. We construct a chain of acceptable quadruples $\langle \Gamma_n, C_n, D_n, \Delta_n \rangle$, with $\Gamma_n \nvdash \varphi$, such that for all $n \in \mathbb{N}$: Γ_{3n+1} and Δ_{3n+1} are prime,

$$\Gamma_{3n+1} \vdash \exists x \psi(x) \;\Rightarrow\; \Gamma_{3n+2} \vdash \psi(e) \text{ for some } e \in C_{3n+2}, \quad \text{and}$$
$$\Delta_{3n+2} \vdash \exists x \psi(x) \;\Rightarrow\; \Delta_{3n+3} \vdash \psi(e) \text{ for some } e \in D_{3n+3}.$$

Set $\Gamma_0 = \Gamma$, $C_0 = C$, $D_0 = D$, and $\Delta_0 = \Delta$. We proceed by induction on $n \in \mathbb{N}$.

Step 3n+1: Suppose $\langle \Gamma_{3n}, C_{3n}, D_{3n}, \Delta_{3n} \rangle$ is acceptable, and $\Gamma_{3n} \nvdash \varphi$. Let **S** be the set of all acceptable quadruples $\langle \Gamma^*, C_{3n}, D_{3n}, \Delta^* \rangle$ such that $\Gamma_{3n} \subseteq \Gamma^*, \Delta_{3n} \subseteq \Delta^*$, and $\Gamma^* \nvdash \varphi$. We define a partial order on **S** by set inclusion: $\langle \Gamma, C_{3n}, D_{3n}, \Delta \rangle \preceq \langle \Gamma', C_{3n}, D_{3n}, \Delta' \rangle$ if and only if $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. It is clear from the definition of acceptable quadruples and compactness that **S** is closed under unions of chains. Thus, by Zorn's Lemma, there is a maximal element $\langle \Gamma_{3n+1}, C_{3n}, D_{3n}, \Delta_{3n+1} \rangle \in \mathbf{S}$. Set $C_{3n+1} = C_{3n}$ and $D_{3n+1} = D_{3n}$.

Suppose $\Gamma_{3n+1} \vdash \psi \vee \theta$. Assume $\Gamma_{3n+1} \cup \{\psi\} \vdash \varphi$ and $\Gamma_{3n+1} \cup \{\theta\} \vdash \varphi$. Then $\Gamma_{3n+1} \vdash (\psi \to \varphi) \wedge (\theta \to \varphi)$. So $\Gamma_{3n+1} \vdash (\psi \vee \theta) \to \varphi$. So $\Gamma_{3n+1} \vdash \varphi$. Contradiction. Thus, without loss of generality, we may suppose $\Gamma_{3n+1} \cup \{\psi\} \nvdash \varphi$. Let

$$\Gamma' = \mathrm{Th}[C_{3n}](\Gamma_{3n+1} \cup \{\psi\}) \quad \text{and} \quad \Delta' = \mathrm{Th}[D_{3n}](\Delta_{3n+1} \cup (\Gamma' \cap \mathcal{A}\mathrm{t}(C_{3n}))).$$

By the acceptability of $\langle \Gamma_{3n+1}, C_{3n}, D_{3n}, \Delta_{3n+1} \rangle$, we have $\Delta_{3n+1} \cap \mathcal{U}(C_{3n}) \subseteq \Gamma_{3n+1} \subseteq \Gamma'$. So by 3.3, the quadruple $\langle \Gamma', C_{3n}, D_{3n}, \Delta' \rangle$ is acceptable, and so is in **S**. By maximality, since $\Gamma_{3n+1} \subseteq \Gamma'$ and $\Delta_{3n+1} \subseteq \Delta'$, we have $\Gamma_{3n+1} = \Gamma'$. Thus, $\Gamma_{3n+1} \vdash \psi$.

Suppose $\Delta_{3n+1} \vdash \psi \vee \theta$. Assume $\Gamma_{3n+1} \cup (\mathrm{Th}[D_{3n}](\Delta_{3n+1} \cup \{\psi\}) \cap \mathcal{U}(C_{3n})) \vdash \varphi$ and $\Gamma_{3n+1} \cup (\mathrm{Th}[D_{3n}](\Delta_{3n+1} \cup \{\theta\}) \cap \mathcal{U}(C_{3n})) \vdash \varphi$. By compactness, since $\mathcal{U}(C_{3n})$ is closed under finite conjunctions, there is a $\rho \in \mathrm{Th}[D_{3n}](\Delta_{3n+1} \cup \{\psi\}) \cap \mathcal{U}(C_{3n})$ and a $\sigma \in \mathrm{Th}[D_{3n}](\Delta_{3n+1} \cup \{\theta\}) \cap \mathcal{U}(C_{3n})$ such that $\Gamma_{3n+1} \cup \{\rho\} \vdash \varphi$ and $\Gamma_{3n+1} \cup \{\sigma\} \vdash \varphi$. So $\Gamma_{3n+1} \vdash (\rho \to \varphi) \wedge (\sigma \to \varphi)$. So $\Gamma_{3n+1} \vdash (\rho \vee \sigma) \to \varphi$. Also, we have $\Delta_{3n+1} \cup \{\psi\} \vdash \rho$ and $\Delta_{3n+1} \cup \{\theta\} \vdash \sigma$. So $\Delta_{3n+1} \vdash (\psi \to \rho) \wedge (\theta \to \sigma)$. So $\Delta_{3n+1} \vdash (\psi \vee \theta) \to (\rho \vee \sigma)$. Thus, $\Delta_{3n+1} \vdash \rho \vee \sigma$. Since $\mathcal{U}(C_{3n})$ is closed under finite disjunctions, we have $\rho \vee \sigma \in \Delta_{3n+1} \cap \mathcal{U}(C_{3n})$. By the acceptability of $\langle \Gamma_{3n+1}, C_{3n}, D_{3n}, \Delta_{3n+1} \rangle$, we have $\rho \vee \sigma \in \Gamma_{3n+1}$. So $\Gamma_{3n+1} \vdash \varphi$. Contradiction. Thus, without loss of generality, we may suppose $\Gamma_{3n+1} \cup (\mathrm{Th}[D_{3n}](\Delta_{3n+1} \cup \{\psi\}) \cap \mathcal{U}(C_{3n})) \nvdash \varphi$. Let

$\Gamma' = \mathrm{Th}[C_{3n}](\Gamma_{3n+1} \cup (\mathrm{Th}[D_{3n}](\Delta_{3n+1} \cup \{\psi\}) \cap \mathcal{U}(C_{3n})))$ and
$\Delta' = \mathrm{Th}[D_{3n}](\Delta_{3n+1} \cup \{\psi\} \cup (\Gamma' \cap \mathcal{A}\mathrm{t}(C_{3n})))$.

By 3.3, the quadruple $\langle \Gamma', C_{3n}, D_{3n}, \Delta' \rangle$ is acceptable, and so is in $\mathbf{S}$. By maximality, since $\Gamma_{3n+1} \subseteq \Gamma'$ and $\Delta_{3n+1} \subseteq \Delta'$, we have $\Delta_{3n+1} = \Delta'$. Thus, $\Delta_{3n+1} \vdash \psi$.

Step 3n+2: Suppose $\langle \Gamma_{3n+1}, C_{3n+1}, D_{3n+1}, \Delta_{3n+1} \rangle$ is acceptable, and $\Gamma_{3n+1} \not\vdash \varphi$. For every sentence $\exists x\psi(x) \in \Gamma_{3n+1}$, let $e_{\exists x\psi(x)}$ be a new constant in E. Let $E' = \{e_{\exists x\psi(x)} : \exists x\psi(x) \in \Gamma_{3n+1}\}$. We pick E' so that $|E \setminus E'| = |E|$. Set $C_{3n+2} = C_{3n+1} \cup E'$ and $D_{3n+2} = D_{3n+1} \cup E'$. Set

$$\Gamma_{3n+2} = \mathrm{Th}[C_{3n+2}](\Gamma_{3n+1} \cup \{\psi(e_{\exists x\psi(x)}) : \exists x\psi(x) \in \Gamma_{3n+1}\}).$$

Note that for every sentence $\exists x\psi(x) \in \Gamma_{3n+1}$, there is an $e \in C_{3n+2}$ such that $\Gamma_{3n+2} \vdash \psi(e)$. Assume $\Gamma_{3n+2} \vdash \varphi$. By compactness, there is a sentence $\theta(\mathbf{e}) := \psi_1(e_1) \wedge ... \wedge \psi_m(e_m)$, with $\mathbf{e} \in E'$ and $\exists x\psi_1(x), ..., \exists x\psi_m(x) \in \Gamma_{3n+1}$, such that $\Gamma_{3n+1} \cup \{\theta(\mathbf{e})\} \vdash \varphi$. So $\Gamma_{3n+1} \vdash \theta(\mathbf{e}) \to \varphi$. Since $\mathbf{e} \notin C_{3n+1}$, we have $\Gamma_{3n+1} \vdash \forall \mathbf{x}(\theta(\mathbf{x}) \to \varphi)$. So $\Gamma_{3n+1} \vdash \exists \mathbf{x}\theta(\mathbf{x}) \to \varphi$. Since $\exists x\psi_1(x), ..., \exists x\psi_m(x) \in \Gamma_{3n+1}$, we have $\exists \mathbf{x}\theta(\mathbf{x}) \in \Gamma_{3n+1}$. So $\Gamma_{3n+1} \vdash \varphi$. Contradiction. Thus, $\Gamma_{3n+2} \not\vdash \varphi$. We claim $\mathrm{Th}[D_{3n+2}](\Delta_{3n+1}) \cap \mathcal{U}(C_{3n+2}) \subseteq \Gamma_{3n+2}$. Suppose $\rho(\mathbf{e}) \in \mathrm{Th}[D_{3n+2}](\Delta_{3n+1}) \cap \mathcal{U}(C_{3n+2})$ with $\mathbf{e} \in E'$. Since $\mathbf{e} \notin D_{3n+1}$, we have $\Delta_{3n+1} \vdash \forall \mathbf{x}\rho(\mathbf{x}) \in \mathcal{U}(C_{3n+1})$. By induction hypothesis, $\Delta_{3n+1} \cap \mathcal{U}(C_{3n+1}) \subseteq \Gamma_{3n+1}$. So $\Gamma_{3n+1} \vdash \forall \mathbf{x}\rho(\mathbf{x})$. So $\Gamma_{3n+2} \vdash \rho(\mathbf{e})$, which proves the claim. Set

$$\Delta_{3n+2} = \mathrm{Th}[D_{3n+2}](\Delta_{3n+1} \cup (\Gamma_{3n+2} \cap \mathcal{A}\mathrm{t}(C_{3n+2}))).$$

By 3.3, $\langle \Gamma_{3n+2}, C_{3n+2}, D_{3n+2}, \Delta_{3n+2} \rangle$ is acceptable.

Step 3n+3: Suppose $\langle \Gamma_{3n+2}, C_{3n+2}, D_{3n+2}, \Delta_{3n+2} \rangle$ is acceptable, and $\Gamma_{3n+2} \not\vdash \varphi$. For every sentence $\exists x\psi(x) \in \Delta_{3n+2}$, let $e_{\exists x\psi(x)}$ be a new constant in E. Let $E' = \{e_{\exists x\psi(x)} : \exists x\psi(x) \in \Delta_{3n+2}\}$. We pick E' so that $|E \setminus E'| = |E|$. Set $C_{3n+3} = C_{3n+2}$ and $D_{3n+3} = D_{3n+2} \cup E'$. Set $\Gamma_{3n+3} = \Gamma_{3n+2}$, and set

$$\Delta_{3n+3} = \mathrm{Th}[D_{3n+3}](\Delta_{3n+2} \cup \{\psi(e_{\exists x\psi(x)}) : \exists x\psi(x) \in \Delta_{3n+2}\}).$$

Note that for every sentence $\exists x\psi(x) \in \Delta_{3n+2}$, there is an $e \in D_{3n+3}$ such that $\Delta_{3n+3} \vdash \psi(e)$. By the acceptability of $\langle \Gamma_{3n+2}, C_{3n+2}, D_{3n+2}, \Delta_{3n+2} \rangle$, we have $\Gamma_{3n+3} \cap \mathcal{A}\mathrm{t}(C_{3n+3}) \subseteq \Delta_{3n+2} \subseteq \Delta_{3n+3}$. Let $\Gamma' = \mathrm{Th}[C_{3n+2}](\Gamma_{3n+2} \cup (\Delta_{3n+3} \cap \mathcal{U}(C_{3n+2})))$. We claim $\Gamma' = \Gamma_{3n+3}$. Suppose $\sigma \in \Gamma'$. By compactness, and since $\mathcal{U}(C_{3n+2})$ is closed under finite conjunctions, there is a $\rho \in \Delta_{3n+3} \cap \mathcal{U}(C_{3n+2})$ such that $\Gamma_{3n+2} \cup \{\rho\} \vdash \sigma$. By compactness again, there is a sentence $\theta(\mathbf{e}) := \psi_1(e_1) \wedge ... \wedge \psi_m(e_m)$, with $\mathbf{e} \in E'$ and $\exists x\psi_1(x), ..., \exists x\psi_m(x) \in \Delta_{3n+2}$, such that $\Delta_{3n+2} \cup \{\theta(\mathbf{e})\} \vdash \rho$. Since $\mathbf{e} \notin D_{3n+2} \supseteq C_{3n+2}$, $\mathbf{e}$ does not appear in Δ_{3n+2} or in ρ. So $\Delta_{3n+2} \cup \{\exists \mathbf{x}\theta(\mathbf{x})\} \vdash \rho$. Since $\exists \mathbf{x}\theta(\mathbf{x}) \in \Delta_{3n+2}$, we have $\Delta_{3n+2} \vdash \rho$. By induction hypothesis, $\rho \in \Delta_{3n+2} \cap \mathcal{U}(C_{3n+2}) \subseteq \Gamma_{3n+2}$. But we also have $\Gamma_{3n+2} \vdash \rho \to \sigma$. So $\Gamma_{3n+2} \vdash \sigma$. So $\Gamma' \subseteq \Gamma_{3n+2} = \Gamma_{3n+3}$, and the claim is proven. It is easy to see that $\Delta_{3n+3} \cap \mathcal{U}(C_{3n+3}) \subseteq \Gamma' = \Gamma_{3n+3}$. So $\langle \Gamma_{3n+3}, C_{3n+3}, D_{3n+3}, \Delta_{3n+3} \rangle$ is acceptable.

This completes the induction on $n \in \mathbb{N}$. Set $\Gamma' = \bigcup \Gamma_n$, $C' = \bigcup C_n$, $D' = \bigcup D_n$, and $\Delta' = \bigcup \Delta_n$. By compactness, Γ' is C'-Henkin prime, Δ' is D'-Henkin prime, and $\Gamma' \not\vdash \varphi$. $\dashv$

It is useful to note that if $\langle \Gamma, C, D, \Delta \rangle$ is an acceptable quadruple in which Γ is already C-Henkin prime, then the construction used in the proof of Proposition 3.4 can easily be modified to yield an acceptable quadruple $\langle \Gamma, C, D', \Delta' \rangle$

such that Δ' is D'-Henkin prime, $\Delta \subseteq \Delta'$, and $D \subseteq D'$. This is accomplished by letting $\Gamma_n = \Gamma$ and $C_n = C$ for all n in the chain of acceptable quadruples, and by skipping step 3n+2.

Proposition 3.5 *Let $\mathcal{L}$ be a first order language, let C and D be sets of constants, and let Γ and Δ be theories such that $\langle \Gamma, C, D, \Delta \rangle$ is acceptable, Γ is C-Henkin prime, and Δ is D-Henkin prime. Then there are rooted Kripke models $\mathfrak{A}$ and $\mathfrak{B}$, with $\mathfrak{A} \subseteq \mathfrak{B}$, such that*

$$\mathfrak{A} \Vdash \varphi \;\Leftrightarrow\; \Gamma \vdash \varphi, \quad \text{for all } \varphi \in \mathcal{L}(C), \quad \text{and}$$
$$\mathfrak{B} \Vdash \varphi \;\Leftrightarrow\; \Delta \vdash \varphi, \quad \text{for all } \varphi \in \mathcal{L}(D).$$

Proof. Let X be a set of new constants such that $|X| \geq |\mathcal{L}(D)|$. Let $\mathbf{C}$ be the following poset category: The nodes of $\mathbf{C}$ are all quadruples $\langle \Gamma', C', D', \Delta' \rangle$ with $\Gamma \subseteq \Gamma'$, $C \subseteq C'$, $D \subseteq D'$, and $\Delta \subseteq \Delta'$, where Γ' is a consistent theory over $\mathcal{L}(C')$, Δ' is a consistent theory over $\mathcal{L}(D')$, $C' \subseteq D' \subseteq D \cup X$, $|D'| \leq |\mathcal{L}(D)|$, and $|X \setminus D'| = |X|$. The order on $\mathbf{C}$ is defined by $\langle \Gamma', C', D', \Delta' \rangle \preceq \langle \Gamma'', C'', D'', \Delta'' \rangle$ if and only if $\Gamma' \subseteq \Gamma''$, $C' \subseteq C''$, $D' \subseteq D''$, and $\Delta' \subseteq \Delta''$. Let $\mathbf{B} \subseteq \mathbf{C}$ be the poset category of all quadruples $\langle \Gamma', C', D', \Delta' \rangle$ in $\mathbf{C}$ such that Γ' is a C'-Henkin prime theory and Δ' is a D'-Henkin prime theory. We define a Kripke model $\mathfrak{B} : \mathbf{B} \to \mathbf{M}(\mathcal{L})$ by:

$$B_{\langle \Gamma', C', D', \Delta' \rangle} = D'/\equiv, \quad \text{where } a \equiv b \text{ if and only if } \Delta' \vdash a = b$$
$$\mathfrak{B}_{\langle \Gamma', C', D', \Delta' \rangle} \models \varphi \;\Leftrightarrow\; \Delta' \vdash \varphi, \quad \text{for all } \varphi \in \mathcal{A}\mathrm{t}(D')$$
$$\mathfrak{B}f : \mathfrak{B}_i \to \mathfrak{B}_j \text{ is defined by } \mathfrak{B}f(a^{\mathfrak{B}_i}) = a^{\mathfrak{B}_j}.$$

Let $\mathbf{A} \subseteq \mathbf{B}$ be the poset category of all acceptable quadruples in $\mathbf{B}$. We define a Kripke model $\mathfrak{A} : \mathbf{A} \to \mathbf{M}(\mathcal{L})$ by:

$$A_{\langle \Gamma', C', D', \Delta' \rangle} = C'/\equiv, \quad \text{where } a \equiv b \text{ if and only if } \Gamma' \vdash a = b$$
$$\mathfrak{A}_{\langle \Gamma', C', D', \Delta' \rangle} \models \varphi \;\Leftrightarrow\; \Gamma' \vdash \varphi, \quad \text{for all } \varphi \in \mathcal{A}\mathrm{t}(C')$$
$$\mathfrak{A}f : \mathfrak{A}_i \to \mathfrak{A}_j \text{ is defined by } \mathfrak{A}f(a^{\mathfrak{A}_i}) = a^{\mathfrak{A}_j}.$$

It is straightforward to verify that $\mathfrak{A}$ and $\mathfrak{B}$ are Kripke models, and that $\mathfrak{A} \subseteq \mathfrak{B}$. Also, $\mathfrak{A}$ and $\mathfrak{B}$ are rooted, with the same root $i_0 = \langle \Gamma, C, D, \Delta \rangle \in |\mathbf{A}| \subseteq |\mathbf{B}|$. We claim:

For all $i = \langle \Gamma', C', D', \Delta' \rangle \in |\mathbf{A}|$ and $\varphi \in \mathcal{L}(C')$, $\quad i \Vdash^{\mathfrak{A}} \varphi \;\Leftrightarrow\; \Gamma' \vdash \varphi$, and
For all $i = \langle \Gamma', C', D', \Delta' \rangle \in |\mathbf{B}|$ and $\varphi \in \mathcal{L}(D')$, $\quad i \Vdash^{\mathfrak{B}} \varphi \;\Leftrightarrow\; \Delta' \vdash \varphi$.

One can prove the second assertion by making only trivial modifications to the proof of 2.2. This proof is left to the reader. We prove the first assertion.

Let $i = \langle \Gamma', C', D', \Delta' \rangle \in |\mathbf{A}|$. The proof is by induction on the complexity of $\varphi \in \mathcal{L}(C')$. We prove only the difficult cases ($\forall$ and $\to$).

Suppose $\varphi := \psi \to \theta$ and $\Gamma' \vdash \psi \to \theta$. Let $j = \langle \Gamma'', C'', D'', \Delta'' \rangle \in |\mathbf{A}|$ be such that $i \preceq j$, and suppose $j \Vdash^{\mathfrak{A}} \psi$. By induction hypothesis, $\Gamma'' \vdash \psi$. Since $\Gamma' \subseteq \Gamma''$, we have $\Gamma'' \vdash \psi \to \theta$. So $\Gamma'' \vdash \theta$. So by induction hypothesis again, $j \Vdash^{\mathfrak{A}} \theta$. Thus, $i \Vdash^{\mathfrak{A}} \psi \to \theta$. Conversely, suppose $\varphi := \psi \to \theta$ and $\Gamma' \nvdash \psi \to \theta$. Then $\Gamma' \cup \{\psi\} \nvdash \theta$. Consider the quadruple $\langle \mathrm{Th}[C'](\Gamma' \cup \{\psi\}), C', D', \Delta' \rangle$. Since $\langle \Gamma', C', D', \Delta' \rangle$ is acceptable, we have $\Delta' \cap \mathcal{U}(C') \subseteq \Gamma' \subseteq \mathrm{Th}[C'](\Gamma' \cup \{\psi\})$. So by 3.3, $\langle \mathrm{Th}[C'](\Gamma' \cup \{\psi\}), C', D', \mathrm{Th}[D'](\Delta' \cup (\mathrm{Th}[C'](\Gamma' \cup \{\psi\}) \cap \mathcal{A}\mathrm{t}(C'))) \rangle$ is acceptable. By 3.4, there is an acceptable quadruple $j = \langle \Gamma'', C'', D'', \Delta'' \rangle \in |\mathbf{A}|$ such that $i \preceq j$ and $\Gamma'' \nvdash \theta$. By induction hypothesis, $j \nVdash^{\mathfrak{A}} \theta$. Since

$\psi \in \mathrm{Th}[C''](\Gamma' \cup \{\psi\}) \subseteq \Gamma''$, we also have by induction hypothesis $j \Vdash^{\mathfrak{A}} \psi$. So $j \nVdash^{\mathfrak{A}} \psi \to \theta$. Thus, $i \nVdash^{\mathfrak{A}} \psi \to \theta$.

Suppose $\varphi := \forall x\psi(x)$ and $\Gamma' \vdash \forall x\psi(x)$. Let $j = \langle \Gamma'', C'', D'', \Delta'' \rangle \in |\mathbf{A}|$ be such that $i \preceq j$, and let $a \in A_j$. Since $\Gamma' \subseteq \Gamma''$, we have $\Gamma'' \vdash \forall x\psi(x)$. So $\Gamma'' \vdash \psi(a)$. By induction hypothesis, $j \Vdash^{\mathfrak{A}} \psi(a)$. Thus, $i \Vdash^{\mathfrak{A}} \forall x\psi(x)$. Conversely, suppose $\varphi := \forall x\psi(x)$ and $\Gamma' \nvdash \forall x\psi(x)$. Let $e \in X$ be a new constant. Then $\Gamma' \nvdash \psi(e)$. Let $C'' = C' \cup \{e\}$, and let $D'' = D' \cup \{e\}$. Let $\Gamma'' = \mathrm{Th}[C''](\Gamma')$, and let $\Delta'' = \mathrm{Th}[D''](\Delta')$. Consider the quadruple $\langle \Gamma'', C'', D'', \Delta'' \rangle$. Let $\theta(\mathbf{c}, e) \in \Delta'' \cap \mathcal{U}(C'')$. Then $\Delta'' \vdash \theta(\mathbf{c}, e)$. So $\Delta' \vdash \forall z\theta(\mathbf{c}, z)$, where $\forall z\theta(\mathbf{c}, z) \in \mathcal{U}(C')$. Since $\langle \Gamma', C', D', \Delta' \rangle$ is acceptable, we have $\forall z\theta(\mathbf{c}, z) \in \Delta' \cap \mathcal{U}(C') \subseteq \Gamma'$. So $\Gamma' \vdash \forall z\theta(\mathbf{c}, z)$. Thus, $\Gamma'' \vdash \theta(\mathbf{c}, e)$. So $\theta(\mathbf{c}, e) \in \Gamma''$. So $\Delta'' \cap \mathcal{U}(C'') \subseteq \Gamma''$. By 3.3, $\langle \Gamma'', C'', D'', \mathrm{Th}[D''](\Delta'' \cup (\Gamma'' \cap \mathcal{A}\mathrm{t}(C''))) \rangle$ is acceptable. By 3.4, there is an acceptable quadruple $j = \langle \Gamma''', C''', D''', \Delta''' \rangle \in |\mathbf{A}|$ such that $i \preceq j$ and $\Gamma''' \nvdash \psi(e)$. By induction hypothesis, $j \nVdash^{\mathfrak{A}} \psi(e)$, where $e \in A_j$. Thus, $i \nVdash^{\mathfrak{A}} \forall x\psi(x)$.

The other cases are easier. The Henkin property will be used in the $\exists$ case, and the prime property will be used in the $\vee$ case.

Since $i_0 = \langle \Gamma, C, D, \Delta \rangle \in |\mathbf{A}| \subseteq |\mathbf{B}|$, we have

$$i_0 \Vdash^{\mathfrak{A}} \varphi \;\Leftrightarrow\; \Gamma \vdash \varphi, \quad \text{for all } \varphi \in \mathcal{L}(C), \quad \text{and}$$
$$i_0 \Vdash^{\mathfrak{B}} \varphi \;\Leftrightarrow\; \Delta \vdash \varphi, \quad \text{for all } \varphi \in \mathcal{L}(D).$$

This completes the proof. ⊣

4 Preservation and Model Consistency

There is a well-known classical preservation theorem which states that a classical theory $\Delta \supseteq \Gamma$ is axiomatizable by universal sentences over Γ if and only if Δ is preserved under Γ-submodels. That is, a classical theory $\Delta \supseteq \Gamma$ is axiomatizable by universal sentences over Γ if and only if for all classical models $\mathfrak{A} \models \Gamma$ and $\mathfrak{B} \models \Delta$, if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \models \Delta$. Our main result is a direct analogue of this theorem, with our notions of universal sentence and Kripke submodel replacing the classical notions. The theorem below holds generally for intuitionistic theories, and in the case where Γ contains the law of excluded middle as an axiom schema (i.e., $\forall \mathbf{x}(\varphi \vee \neg\varphi)$ for all formulas φ and variables $\mathbf{x}$), it implies the classical theorem.

Theorem 4.1 *Let $\Gamma \subseteq \Delta$ be intuitionistic theories over a language $\mathcal{L}$. Then Δ is axiomatizable by universal sentences over Γ if and only if Δ is preserved under Γ-Kripke submodels.*

Proof. ($\Rightarrow$) See 3.1.

($\Leftarrow$) Suppose that Δ is not axiomatizable by universal sentences over Γ. Consider the quadruple $\langle \mathrm{Th}(\Gamma \cup (\Delta \cap \mathcal{U})), \varnothing, \varnothing, \Delta \rangle$. It is obvious that this quadruple is acceptable. Also, since Δ is not axiomatizable by universal sentences over Γ, there is a sentence $\varphi \in \Delta$ such that $\mathrm{Th}(\Gamma \cup (\Delta \cap \mathcal{U})) \nvdash \varphi$. So by 3.4, there is an acceptable quadruple $\langle \Gamma', C', D', \Delta' \rangle$ such that Γ' is C'-Henkin prime, Δ' is D'-Henkin prime, $\mathrm{Th}(\Gamma \cup (\Delta \cap \mathcal{U})) \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ and $\Gamma' \nvdash \varphi$. By 3.5, there are rooted Kripke models $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A} \subseteq \mathfrak{B}$ such that $\mathfrak{A} \Vdash \psi$ if and only if $\Gamma' \vdash \psi$ and $\mathfrak{B} \Vdash \psi$ if and only if $\Delta' \vdash \psi$. So $\mathfrak{A} \nVdash \varphi$. Thus, we have $\mathfrak{A} \Vdash \Gamma \subseteq \Gamma'$, $\mathfrak{A} \nVdash \Delta$, $\mathfrak{B} \Vdash \Delta \subseteq \Delta'$, and $\mathfrak{A} \subseteq \mathfrak{B}$. So Δ is not preserved under Γ-Kripke submodels. ⊣

Note that the classical theorem is an easy consequence of Theorem 4.1 and the following two lemmas. As in [5], we denote the set of classically valid sentences by CQC.

Lemma 4.2 *Let $\mathfrak{A}$ be a Kripke model over a language $\mathcal{L}$. Then $\mathfrak{A} \Vdash$ CQC if and only if for all $i \in |\mathbf{A}|$ and all sentences $\varphi \in \mathcal{L}(A_i)$, $i \Vdash^{\mathfrak{A}} \varphi \Leftrightarrow \mathfrak{A}_i \models \varphi$.*

Proof. Left to the reader. ⊣

Lemma 4.3 *Let Γ and Δ be theories such that CQC $\subseteq \Gamma \subseteq \Delta$. Then Δ is preserved under Γ-Kripke submodels if and only if Δ is preserved under classical Γ-submodels.*

Proof. ($\Rightarrow$) Since a classical model can be viewed as a one-node Kripke model, this direction is obvious.

($\Leftarrow$) Suppose CQC $\subseteq \Gamma \subseteq \Delta$, and Δ is preserved under classical Γ-submodels. Let $\mathfrak{A} \Vdash \Gamma$ and $\mathfrak{B} \Vdash \Delta$ be Kripke models such that $\mathfrak{A} \subseteq \mathfrak{B}$. Since $\mathfrak{A} \Vdash$ CQC and $\mathfrak{B} \Vdash$ CQC, we have $\mathfrak{A}_i \models \Gamma$ for all $i \in |\mathbf{A}|$ and $\mathfrak{B}_j \models \Delta$ for all $j \in |\mathbf{B}|$. Since $\mathfrak{A}_i \subseteq \mathfrak{B}_i$ for all $i \in |\mathbf{A}|$, we have by assumption $\mathfrak{A}_i \models \Delta$ for all $i \in |\mathbf{A}|$. So, since $\mathfrak{A} \Vdash$ CQC, $i \Vdash^{\mathfrak{A}} \Delta$ for all $i \in |\mathbf{A}|$. So $\mathfrak{A} \Vdash \Delta$. So Δ is preserved under Γ-Kripke submodels. ⊣

There is another classical result involving models and universal theories, which states that if Γ and Δ are classical theories, then the universal fragment of Δ is contained in the universal fragment of Γ if and only if every model of Γ is contained in a model of Δ, in which case Δ is said to be *model-consistent* relative to Γ. So two theories are model-consistent relative to each other if and only if they have the same universal fragment. To state our next result, we need a definition. In the following, we will write $(\mathfrak{A}, r)$ to denote a rooted Kripke model with root r.

Definition 4.4 *Let $(\mathfrak{A}, r)$ be a rooted Kripke model. Then $\mathrm{Th}(\mathfrak{A}, r) = \{\varphi \in \mathcal{L}(A_r) : \mathfrak{A} \Vdash \varphi\}$. If $(\mathfrak{A}, r)$ and $(\mathfrak{B}, s)$ are rooted Kripke models, we say that $(\mathfrak{A}, r)$ and $(\mathfrak{B}, s)$ are* **strongly equivalent***, $(\mathfrak{A}, r) \sim (\mathfrak{B}, s)$, if $\mathrm{Th}(\mathfrak{A}, r) = \mathrm{Th}(\mathfrak{B}, s)$.*

If $(\mathfrak{A}, r)$ and $(\mathfrak{B}, s)$ are rooted Kripke models and $\mathrm{Th}(\mathfrak{A}, r) = \mathrm{Th}(\mathfrak{B}, s)$, then, by the definition of forcing, $\mathrm{Th}(\mathfrak{A}_r) \cap \mathcal{A}\mathrm{t}(A_r) = \mathrm{Th}(\mathfrak{B}_s) \cap \mathcal{A}\mathrm{t}(B_s)$. So if $(\mathfrak{A}, r) \sim (\mathfrak{B}, s)$, then the root structures $\mathfrak{A}_r$ and $\mathfrak{B}_s$ are isomorphic in the classical sense.

We prove an analogue of the classical model-consistency result for intuitionistic logic, using our notions of submodel and universal sentence: If Γ and Δ are intuitionistic theories, then the universal fragment of Δ is contained in the universal fragment of Γ if and only if every rooted Kripke model of Γ is strongly equivalent to a submodel of a rooted Kripke model of Δ. First we need the following lemma, which is a converse of Proposition 2.2.

Lemma 4.5 *Let $(\mathfrak{A}, r)$ be a rooted Kripke model. Then $\mathrm{Th}(\mathfrak{A}, r)$ is an A_r-Henkin prime theory.*

Proof. This follows directly from the definition of forcing in a rooted Kripke model. (See [2], Section 5.4.) ⊣

Theorem 4.6 *Let Γ and Δ be intuitionistic theories over a language $\mathcal{L}$. Then $\Delta \cap \mathcal{U} \subseteq \Gamma$ if and only if for every rooted Kripke model $(\mathfrak{A}, r) \Vdash \Gamma$ there are rooted Kripke models $(\mathfrak{A}', r')$ and $(\mathfrak{B}, s) \Vdash \Delta$ such that $(\mathfrak{A}, r) \sim (\mathfrak{A}', r') \subseteq (\mathfrak{B}, s)$.*

Proof. ($\Rightarrow$) Suppose $\Delta \cap \mathcal{U} \subseteq \Gamma$. Let $(\mathfrak{A}, r) \Vdash \Gamma$ be a rooted Kripke model. Then $\Gamma' = \mathrm{Th}(\mathfrak{A}, r)$ is an A_r-Henkin prime theory over $\mathcal{L}(A_r)$, and $\mathrm{Th}[A_r](\Gamma) \subseteq \Gamma'$. By 3.3, since $\mathrm{Th}[A_r](\Delta) \cap \mathcal{U}(A_r) \subseteq \mathrm{Th}[A_r](\Gamma) \subseteq \Gamma'$, the quadruple $\langle \Gamma', A_r, A_r, \mathrm{Th}[A_r](\Delta \cup (\Gamma' \cap \mathcal{A}\mathrm{t}(A_r))) \rangle$ is acceptable. Since Γ' is already A_r-Henkin prime, the modified construction discussed after the proof of 3.4 yields an acceptable quadruple $\langle \Gamma', A_r, D', \Delta' \rangle$, where $A_r \subseteq D'$, $\mathrm{Th}[A_r](\Delta \cup (\Gamma' \cap \mathcal{A}\mathrm{t}(A_r))) \subseteq \Delta'$, and Δ' is D'-Henkin prime. By 3.5, there are rooted Kripke models $(\mathfrak{A}', r') \subseteq (\mathfrak{B}, s)$ such that $\mathrm{Th}(\mathfrak{A}', r') = \Gamma' = \mathrm{Th}(\mathfrak{A}, r)$, and $(\mathfrak{B}, s) \Vdash \Delta' \supseteq \Delta$.

($\Leftarrow$) Let $(\mathfrak{A}, r) \Vdash \Gamma$ be a rooted Kripke model. By hypothesis, there are rooted Kripke models $(\mathfrak{A}', r')$ and $(\mathfrak{B}, s) \Vdash \Delta$ such that $(\mathfrak{A}, r) \sim (\mathfrak{A}', r') \subseteq (\mathfrak{B}, s)$. Since universal sentences are preserved under Kripke submodels, we have $(\mathfrak{A}', r') \Vdash \Delta \cap \mathcal{U}$. So also $(\mathfrak{A}, r) \Vdash \Delta \cap \mathcal{U}$. Thus, by the completeness of rooted Kripke models for intuitionistic logic, $\Gamma \vdash \Delta \cap \mathcal{U}$. $\dashv$

In the case where every Kripke model consists of a single node with only the identity morphism, Theorem 4.6 reduces to the classical result for model-consistent theories.

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