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Basic Predicate Calculus

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Abstract We establish a completeness theorem for first-order basic predicate logic BQC, a proper subsystem of intuitionistic predicate logic IQC, using Kripke models with transitive underlying frames. We develop the notion of functional well-formed theory as the right notion of theory over BQC for which strong completeness theorems are possible. We also derive the undecidability of basic arithmetic, the basic logic equivalent of intuitionistic Heyting Arithmetic and classical Peano Arithmetic.

1 Introduction Basic Predicate Calculus (BQC) was motivated by a revision of the Brouwer-Heyting-Kolmogorov proof interpretation (Ruitenburg [5], [6]). Before an actual axiomatization for BQC was attempted, a class of models for which BQC should satisfy a completeness theorem was established: Kripke models as for Intuitionistic Predicate Calculus (IQC), except that the order on the underlying set of nodes is transitive but not necessarily reflexive. This class naturally generalizes the class of models for Basic Propositional Calculus (BPC), for which axiomatizations and completeness were established (Ardeshir and Ruitenburg [1], Visser [8]). Unfortunately, the original axiomatizations of BQC were wrong. The versions in [5], [6], and Ardeshir [2] refer to a predicate calculus that in reality is a proper sublogic between the intended BQC, described in this paper, and IQC. In this paper we correct the mistake by formulating a new axiomatization and by providing a first detailed proof of the completeness theorem. As BQC is essentially weaker than IQC, “standard” or “expected” results must be presented with extra detail.

We also explore the question of what is a good definition of theory. A good theory over BQC should have a proper balance between formulas $\forall\mathbf{x}(A \to B)$ and sequents $A \Longrightarrow B$. In [1] the balance for BPC is struck with formalization, on the one hand, and with faithfulness on the other. Here both will be generalized in two ways: first, by allowing for theories with rule as well as sequent axioms and second, by extending both notions to BQC. A very general notion of a theory is introduced from which a stricter version of a functional, well-formed theory that is adequate to our purposes is derived. One example of this is the theory of Basic Arithmetic, that is,

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the basic logic equivalent of Peano Arithmetic and Heyting Arithmetic. Functional, well-formed theories are essentially more general than sequent theories as discussed in [1] and [6].

We present *two* completeness theorems for BQC. In Section 3 a quick overview of the first completeness theorem's basic ideas and the structure of its proof is presented. The second, stronger, completeness theorem is discussed in Section 5 with more detail, because it is sensitive to details. Many results in [1] involving BPC have immediate generalizations to BQC. We attempt to restrict our attention to those generalizations that require a significantly different formulation or a significantly different proof. We apply our results to the theory BA of Basic Arithmetic and, among other things, we establish its undecidability.

There is the issue of the name "basic logic", which is a less than optimal choice. We hope one day a better name comes along. There is motivation for using an alternative name, but this name—geometric logic—is already taken.

2 Axiomatization and rules The language for BQC is not the same as the usual one for IQC. The reason for the alternative choice is the following. Over both IQC and BQC a formula of the form $\forall x A$ corresponds with a formula of the form $Ex \to A$, where E is the extent operator of Heyting ([3], [4]) and Scott [7]. Similarly the formula $\forall x \forall y A$ corresponds with a formula of the form $Ex \to (Ey \to A)$ which is, at least over IQC, equivalent to $Ex \wedge Ey \to A$. Informally, over IQC, this equivalence is an excuse for us to use the abbreviation $\forall xy A$ for the original formula $\forall x \forall y A$. However, over BQC the formula $Ex \to (Ey \to A)$ may be essentially weaker than $Ex \wedge Ey \to A$. So over BQC we prefer to distinguish between $\forall x \forall y A$ and $\forall xy A$. A simple further extension is also to allow universal quantifications that correspond with expressions of the form $Ex \wedge Ey \wedge B \to A$. So over BQC we have more involved universal quantification expressions for these of the form $\forall xy(B \to A)$. More generally, we admit expressions of the form $\forall \mathbf{x}(B \to A)$, where $\mathbf{x}$ is a finite sequence of variables. We conventionally use small boldface letters to represent sequences of terms, and regular small letters for single terms. A convenient side effect of having these more general expressions is, that they allow us to syntactically redefine implication as a special case of universal quantification: if $\mathbf{x}$ is an empty sequence of variables, then we may write $B \to A$ for $\forall \mathbf{x}(B \to A)$. For existential quantification no such problems occur over BQC. So, as is usual over IQC, we may occasionally write $\exists \mathbf{x} A$ as short for $\exists x_1 \exists x_2 \ldots \exists x_n A$.

The language of BQC has a set of predicate symbols of varying finite arity, a set of function symbols of varying finite arity, a countably infinite set of variables, parentheses, logical constants $\top$ and $\bot$, and the logical connectives $\wedge$, $\vee$, $\exists$, and $\forall$. Constant symbols occur as function symbols of arity 0. We usually include the binary predicate $=$ for equality. Terms, atomic formulas, and formulas are defined as usual, except that for universal quantification we have the more elaborate rule: if A and B are formulas, and $\mathbf{x}$ is a finite sequence of variables, then $\forall \mathbf{x}(A \to B)$ is also a formula. Free variables are defined in the obvious way. A *sentence* is a formula without free variables. A *closed term* is a term without free variables.

We may write $A \to B$ for $\forall(A \to B)$, that is, implication is universal quantification with an empty sequence of variables. Additionally, we employ the usual abbreviations of $\neg A$ and $A \longleftrightarrow B$ as short for $A \to \bot$ and $(A \to B) \wedge (B \to A)$, respectively. (Note that for obvious reasons we can no longer use the expression $\forall \mathbf{x} A$ as a short for $\forall \mathbf{x}(\top \to A)$; see the earlier notation of [6].) The set of quantifier-free formulas is defined as usual, except that it is also closed under universal quantifications with empty sequence of variables, that is, closed under implication. Stated differently, the expression 'quantifier-free' is a misnomer for 'quantifier-variable-free'. Given a sequence of variables $\mathbf{x}$ without repetitions, we write $s^{\mathbf{x}}_{\mathbf{t}}$ for the term and $A^{\mathbf{x}}_{\mathbf{t}}$ for the formulas that result from substituting the terms of $\mathbf{t}$ for all free occurrences of the variables of $\mathbf{x}$ in the term s or the formula A. Note that $A^{\mathbf{x},\mathbf{y}}_{\mathbf{t},\mathbf{u}}$ need not be the same as $(A^{\mathbf{x}}_{\mathbf{t}})^{\mathbf{y}}_{\mathbf{u}}$, similarly for terms. We occasionally borrow this notation for substitution of terms for constant symbols, with the obvious meaning.

There are several possible ways to axiomatize BQC. Here we prefer a version using axiom sequents and rules. For the rules a single horizontal line means that if the sequents above the line hold, then so do the ones below the line. A double line means the same, but in both directions. The BQC axioms that don't involve the quantifiers are essentially those for a distributive lattice with top and bottom. So BQC satisfies all substitution instances of

$$A \Longrightarrow A$$

$$\frac{A \Longrightarrow B \quad B \Longrightarrow C}{A \Longrightarrow C}$$

$$A \Longrightarrow \top \qquad\qquad \bot \Longrightarrow A$$

$$\frac{A \Longrightarrow B \quad A \Longrightarrow C}{\overline{A \Longrightarrow B \wedge C}} \qquad \frac{B \Longrightarrow A \quad C \Longrightarrow A}{\overline{B \vee C \Longrightarrow A}}$$

$$A \wedge (B \vee C) \Longrightarrow (A \wedge B) \vee (A \wedge C).$$

Variable substitution and existential quantification are without surprises:

$$\frac{A \Longrightarrow B}{A^{\mathbf{x}}_{\mathbf{t}} \Longrightarrow B^{\mathbf{x}}_{\mathbf{t}}}\ ,$$

where no variable in the finite sequence of terms $\mathbf{t}$ is bound by a quantifier in the denominator.

$$\frac{B \Longrightarrow A}{\overline{\exists x B \Longrightarrow A}}\ ;$$

and

$$A \wedge \exists x B \Longrightarrow \exists x(A \wedge B)$$

with x not free in A in either the rule or the axiom schema. We usually have equality as part of our language. In such cases we must add the schemas

$$\top \Longrightarrow x = x\,;$$

and

$$x = y \wedge A \Longrightarrow A^{x}_{y}\,,$$

where A is atomic. This completes our list of schemas that don't involve $\forall$. The fragment that doesn't involve universal quantifiers is called *geometric logic*, and sequents that don't involve $\forall$ are called *geometric sequents*. The axioms and rules for $\forall$ form an essentially straightforward generalization of the BPC proposition logical axioms and rules for $\to$ [1, 5, 6]:

$$\frac{A \wedge B \Longrightarrow C}{A \Longrightarrow \forall \mathbf{x}(B \to C)}\ ,$$

where no variable in $\mathbf{x}$ is free in A.

$$\forall \mathbf{x}(A \to B) \wedge \forall \mathbf{x}(B \to C) \Longrightarrow \forall \mathbf{x}(A \to C)\,;$$

$$\forall \mathbf{x}(A \to B) \wedge \forall \mathbf{x}(A \to C) \Longrightarrow \forall \mathbf{x}(A \to (B \wedge C))\,;$$

$$\forall \mathbf{x}(B \to A) \wedge \forall \mathbf{x}(C \to A) \Longrightarrow \forall \mathbf{x}((B \vee C) \to A)\,;$$

$$\forall \mathbf{x}(A \to B) \Longrightarrow \forall \mathbf{x}(A^{\mathbf{x}}_{\mathbf{t}} \to B^{\mathbf{x}}_{\mathbf{t}})\,,$$

where no variable in the sequence of terms $\mathbf{t}$ is bound by a quantifier of A or B; and

$$\forall \mathbf{x}(A \to B) \Longrightarrow \forall \mathbf{y}(A \to B)\,,$$

where no variable in $\mathbf{y}$ is free on the left hand side. The schema $\forall \mathbf{x}(A \to B) \Longrightarrow A \to B$ is a special case, and so is the schema $A \to B \Longrightarrow \forall \mathbf{x}(A \to B)$ if no variable in $\mathbf{x}$ is free in $A \to B$.

$$\forall \mathbf{y}x(B \to A) \Longrightarrow \forall \mathbf{y}(\exists x B \to A)$$

where x is not free in A. This completes the axiomatization of BQC.

We write $A \Longleftrightarrow B$ as short for $A \Longrightarrow B$ plus $B \Longrightarrow A$, and often $\Longrightarrow A$, or even A, for $\top \Longrightarrow A$. Let Γ be a set of sequents and rules. We say Γ *entails*, or *proves*, $A \Longrightarrow B$, written $\Gamma \vdash A \Longrightarrow B$, when $A \Longrightarrow B$ can be obtained, after finitely many applications of the BQC rules and the rules of Γ, from the BQC axiom sequents plus the axiom sequents of Γ. Similarly, Γ *entails*, or *proves*, the rule

$$R = \frac{A_1 \Longrightarrow B_1 \quad \ldots \quad A_n \Longrightarrow B_n}{A \Longrightarrow B}\ ,$$

written $\Gamma \vdash R$, when $\Gamma \cup \{A_1 \Longrightarrow B_1, \ldots, A_n \Longrightarrow B_n\} \vdash A \Longrightarrow B$. We usually write $\Gamma \vdash A$ as short for $\Gamma \vdash \Longrightarrow A$; Γ is *consistent* when $\Gamma \nvdash \bot$. A *theory* is a set of sequents and rules closed under derivability. A theory is axiomatizable by a set Γ if it equals the closure of Γ under derivability. A theory is a *sequent theory* if it is axiomatizable by a set of sequents. More generally, Γ' is a *sequent theory over* Γ, or a *sequent theory extension of* Γ, if Γ' is axiomatizable by Γ plus a set of sequents.

3 Kripke models We first prove the completeness theorem for BQC in a special case where, among other things, the only function symbols of the language are 0-ary, that is, are constant symbols. In Section 5 we will consider the strong completeness theorem involving more general theories and involving languages with function symbols. In *this* section it is our purpose to get a quick insight into a completeness theorem for Kripke models.

A *Kripke structure* is a tuple $\mathbf{D} = \langle W^{\mathbf{D}}, \prec^{\mathbf{D}}, D^{\mathbf{D}} \rangle$, where $W^{\mathbf{D}} = W$ is a nonempty set of *nodes*, or *worlds*, with a binary transitive relation $\prec^{\mathbf{D}} = \prec$. We write $\preceq$ for the reflexive closure of $\prec$, and $\succ$ and $\succeq$ for the converse relations of $\prec$ and $\preceq$ respectively. Additionally, $D^{\mathbf{D}} = D$ is a functor from the category $(W, \preceq)$ to the category of sets, that is, to every node α we assign a set $D\alpha$, and to every pair $\alpha \preceq \beta$ a map D^α_β, such that

1. for all α, D^α_α is the identity on $D\alpha$; and
2. for all $\alpha \preceq \beta \preceq \gamma$, $D^\alpha_\gamma = D^\beta_\gamma D^\alpha_\beta$.

Given a Kripke structure $\mathbf{D}$, we define finite powers $\mathbf{D}^n$ by setting $W^n = W$; by setting $\prec^n = \prec$; and by choosing a new functor D^n such that $D^n\alpha = (D\alpha)^n$ for all α and $(D^n)^\alpha_\beta = (D^\alpha_\beta)^n$. In particular, $\mathbf{D}^0$ assigns singleton sets to all nodes $\alpha \in W$, and assigns the unique maps as D^α_β between them, whenever $\alpha \preceq \beta$. Kripke structures over a fixed transitive set $(W, \prec)$ form a functor category in the usual way. It is, in fact, the category of presheaves over $(W, \preceq)$, hence a topos. The transitive set $(W, \prec)$ enables us to recognize additional structure that permits us to have more general interpretations for $\forall$, and thus also for $\to$, than is usual for topos theory with IQC. In the presheaf category, $\mathbf{D}^n$ is the n-fold category-theoretic product.

A Kripke structure is *inhabited* if all sets $D\alpha$ are nonempty. A *Kripke model* is a tuple $\mathbf{K} = \langle \mathbf{D}^{\mathbf{K}}, I^{\mathbf{K}} \rangle$, where $\mathbf{D}^{\mathbf{K}} = \mathbf{D}$ is an inhabited Kripke structure. Moreover, I assigns to each n-ary predicate symbol P a substructure $I(P) = R_P$ of $\mathbf{D}^n$ in the presheaf category over $(W, \preceq)$. So for all α we have a subset $R_P\alpha \subseteq (D\alpha)^n$, and for each pair $\alpha \preceq \beta$ a map $(R_P)^\alpha_\beta : R_P\alpha \to R_P\beta$ that is the restriction of $(D^\alpha_\beta)^n$ to $R_P\alpha$. To each constant symbol c, I assigns a collection of elements $(I(c)_\alpha \in D\alpha)_{\alpha \in W}$ such that $D^\alpha_\beta I(c)_\alpha = I(c)_\beta$ whenever $\alpha \prec \beta$. The interpretation of a constant symbol is essentially the same as a natural transformation $I : \mathbf{1} \to \mathbf{D}$ from the singleton presheaf $\mathbf{1}$ to $\mathbf{D}$ in the presheaf category over $(W, \preceq)$. If the language includes the equality symbol $=$, then $I(=)$ is assigned to the diagonal substructure of $\mathbf{D}^2$, that is, to the usual equality relation in the presheaf category over $(W, \preceq)$.

For each node α we form an extended language $\mathcal{L}[C\alpha]$ of the original language $\mathcal{L}$ by adding a set of constant symbols $C\alpha \cong D\alpha$. The sets of new constants are chosen such that $C\alpha \cap C\beta = \varnothing$ whenever $\alpha \neq \beta$. We sometimes write A_α for a formula that may contain constant symbols from $C\alpha$. For each α we have a map I_α on the set of constant symbols of $\mathcal{L}[C\alpha]$ which assigns to each constant symbol c of $\mathcal{L}$, the constant $I(c)_\alpha$, and to each new constant symbol $c_\alpha \in C\alpha$ its corresponding element $I_\alpha(c_\alpha) = d_\alpha \in D\alpha$. Given $A = A_\alpha$ and $\beta \succ \alpha$, we write A_β for the formula over $\mathcal{L}[C\beta]$, obtained from A_α by replacing each constant symbol c_α in A_α from $C\alpha$ by its corresponding constant symbol $c_\beta \in I^{-1}_\beta D^\alpha_\beta I_\alpha(c_\alpha)$ from $C\beta$.

Let $\mathbf{c} = (c_1, \ldots, c_n)$ be a sequence of n constant symbols of $\mathcal{L}[C\alpha]$, and let P be an n-ary predicate. Then we write $\alpha \Vdash P(\mathbf{c})$ if $I_\alpha(\mathbf{c}) = (I_\alpha(c_1), \ldots, I_\alpha(c_n)) \in R_P\alpha \subseteq (D\alpha)^n$. The relation $\Vdash$ is uniquely extended to all sentences by the inductive definition

$\alpha \Vdash \top$;
$\alpha \Vdash A \wedge B$ if and only if $\alpha \Vdash A$ and $\alpha \Vdash B$;
$\alpha \Vdash A \vee B$ if and only if $\alpha \Vdash A$ or $\alpha \Vdash B$;
$\alpha \Vdash \exists x A$ if and only if there exist $c \in C\alpha$ such that $\alpha \Vdash A^x_c$; and

$\alpha \Vdash \forall \mathbf{x}(A_\alpha \to B_\alpha)$ if and only if for all $\beta \succ \alpha$ and $\mathbf{c} \in (C\beta)^n$, $\beta \Vdash (A_\beta)^{\mathbf{x}}_{\mathbf{c}}$ implies $\beta \Vdash (B_\beta)^{\mathbf{x}}_{\mathbf{c}}$.

An easy proof by induction on the complexity of sentences shows that if $\beta \succeq \alpha \Vdash A_\alpha$, then $\beta \Vdash A_\beta$. So we can extend the relation $\Vdash$ in a natural way to all formulas by

$\alpha \Vdash A_\alpha$ if and only if for all $\beta \succeq \alpha$ and $\mathbf{c} \in (C\beta)^n$ we have $\beta \Vdash (A_\beta)^{\mathbf{x}}_{\mathbf{c}}$,

where $\mathbf{x} = (x_1, \ldots, x_n)$ includes all free variables of A_α. Since implication is defined in terms of universal quantification,

$\alpha \Vdash A_\alpha \to B_\alpha$ if and only if for all $\beta \succ \alpha$, $\beta \Vdash A_\beta$ implies $\beta \Vdash B_\beta$,

for all sentences $A_\alpha \to B_\alpha$. For formulas $A_\alpha \to B_\alpha$ where $\mathbf{x}$ includes all free variables, this generalizes to

$\alpha \Vdash A_\alpha \to B_\alpha$ if and only if for all $\gamma \succ \beta \succeq \alpha$ and $\mathbf{c} \in (C\beta)^n$, $\gamma \Vdash ((A_\beta)^{\mathbf{x}}_{\mathbf{c}})_\gamma$ implies $\gamma \Vdash ((B_\beta)^{\mathbf{x}}_{\mathbf{c}})_\gamma$.

We extend $\Vdash$ to all sequents by

$\alpha \Vdash A_\alpha \Longrightarrow B_\alpha$ if and only if for all $\beta \succeq \alpha$ and $\mathbf{c} \in (C\beta)^n$, $\beta \Vdash (A_\beta)^{\mathbf{x}}_{\mathbf{c}}$ implies $\beta \Vdash (B_\beta)^{\mathbf{x}}_{\mathbf{c}}$.

Note that $\alpha \Vdash A$ if and only if $\alpha \Vdash \Longrightarrow A$. Extend $\Vdash$ to rules as follows. Let R be the rule

$$\frac{(A_1)_\alpha \Longrightarrow (B_1)_\alpha \quad \ldots \quad (A_n)_\alpha \Longrightarrow (B_n)_\alpha}{A_\alpha \Longrightarrow B_\alpha}\ .$$

Then $\alpha \Vdash R$ if and only if for all $\beta \succeq \alpha$, if $\beta \Vdash (A_i)_\beta \Longrightarrow (B_i)_\beta$ for all $i \leq n$, then $\beta \Vdash A_\beta \Longrightarrow B_\beta$. For all formulas A we obviously have

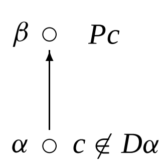
$$\alpha \Vdash x = y \wedge A \Longrightarrow A^x_y\ .$$

A model $\mathbf{K}$ *satisfies* a sequent γ, written $\mathbf{K} \models \gamma$, if and only if $\alpha \Vdash \gamma$ for all nodes $\alpha \in W$: similarly for rules. For sets Γ we write $\mathbf{K} \models \Gamma$ if $\mathbf{K}$ satisfies all rules and sequents of Γ. We write $\Gamma \models \gamma$ if for all models $\mathbf{K}$, if $\mathbf{K} \models \Gamma$, then $\mathbf{K} \models \gamma$: similarly for rules.

Proposition 3.1 (Soundness) *Let Γ be a set of sequents and rules, and γ be a sequent. Then $\Gamma \vdash \gamma$ implies $\Gamma \models \gamma$. If R is a rule such that $\Gamma \vdash R$, then $\Gamma \models R$.*

Proof: Standard. See the proof of Proposition 5.3. □

The following is a counterexample showing that $A \to B \nvdash \forall \mathbf{x}(A \to B)$. Let $\mathbf{K}$ be a Kripke model with two irreflexive nodes $\alpha \prec \beta$ as in the diagram below (open circles indicate irreflexive nodes, filled-in circles indicate reflexive nodes), and D^α_β an inclusion map such that $c \in D\beta \setminus D\alpha$. To simplify notation, we write d as short for $I^{-1}_\gamma(d)$, for all nodes γ and $d \in D\gamma$. Let P be a unary predicate such that $\beta \Vdash Pc$, and $\gamma \nVdash Pd$ for all other combinations of nodes γ and elements d.



Then $\mathbf{K} \models Px \to \bot$, but $\mathbf{K} \not\models \forall x(Px \to \bot)$. Apply soundness. It is just as easy to show that $\top \to Px \nvdash \forall x(\top \to Px)$. In general, if A and B are such that no free variables from among the x_i occur free in A or B, then BQC satisfies $A \to B \vdash \forall \mathbf{x}(A \to B)$.

In the remainder of this section we restrict ourselves to sets of sentences over countable languages without function symbols except constant ones. Many results in this section involving completeness can fairly easily be generalized by modifying some lemma or proposition. Examples are adding function symbols or allowing for noncountable languages. But we don't bother in this section, as we will consider an altogether more general setting in Sections 4 and 5.

Our purpose is to construct sufficiently many Kripke models with which we can prove an easy completeness theorem. Let $\mathcal{L}$ be a countable language with equality, and C be a set of new constant symbols of cardinality continuum. We consider all sets of sentences Γ that are contained in some $\mathcal{L}[D]$, a language obtained from $\mathcal{L}$ by adding some countable subset $D \subset C$. The continuum cardinality of C guarantees that we can expand the (countable) collection of constant symbols in the proofs below. A set Γ is called *deductively closed in language* $\mathcal{L}[D]$, if $\Gamma \subseteq \mathcal{L}[D]$ and, additionally, if $\Gamma \vdash A$ implies $A \in \Gamma$, for all sentences $A \in \mathcal{L}[D]$. Once a set Γ is deductively closed in some language, then this language and its set of constant symbols D can be recovered from the set Γ alone. Therefore we will often write about deductively closed sets of sentences without reference to their languages.

Let $D \subseteq C$ be a countable set of constant symbols. A deductively closed set of sentences $\Gamma \subseteq \mathcal{L}[D]$ is *D-saturated* if

1. Γ is consistent;
2. $A \vee B \in \Gamma$ implies $A \in \Gamma$ or $B \in \Gamma$, for all sentences A and B; and
3. $\exists x A \in \Gamma$ implies $A^x_d \in \Gamma$ for some constant symbol $d \in D$, for all sentences $\exists x A$.

As transitive relation on the collection of deductively closed countable sets of sentences we set $\Gamma \prec \Delta$ if $\forall \mathbf{x}(A \to B) \in \Gamma$ and $A^{\mathbf{x}}_{\mathbf{c}} \in \Delta$ imply $B^{\mathbf{x}}_{\mathbf{c}} \in \Delta$, for all $\forall \mathbf{x}(A \to B)$ and $\mathbf{c}$. So, in particular, $\Gamma \subseteq \Delta$. The relation $\prec$ is easily extended to all countable sets of sentences by replacing occurrences of the form $A \in \Gamma$ by $\Gamma \vdash A$ in the definition above, and so on. Let $\mathrm{Cl}(\Gamma)$ denote the deductive closure of Γ. Then $\Gamma \prec \Delta$ if $\mathrm{Cl}(\Gamma) \prec \mathrm{Cl}(\Delta)$.

Given sets of sentences $\Gamma \subseteq \Delta$, define $\Delta_\Gamma = \{B^{\mathbf{x}}_{\mathbf{c}} \mid \Gamma \vdash \forall \mathbf{x}(A \to B)$ a sentence, and $A^{\mathbf{x}}_{\mathbf{c}} \in \mathrm{Cl}(\Delta)\}$. It is implicit in this definition that Δ_Γ does not introduce constant symbols that don't already occur in Δ. If $\Delta_\Gamma \vdash A$, then $(\Delta')_{\Gamma'} \vdash A$ for some finite $\Delta' \subseteq \Delta$ and $\Gamma' \subseteq \Gamma \cap \Delta'$.

Proposition 3.2 *Let $\Gamma \subseteq \Delta$ be sets of sentences. Then Δ_Γ is deductively closed such that*

1. *$\Gamma \prec \Delta_\Gamma$; and*
2. *$\Gamma \prec \Phi \supseteq \Delta$ implies $\Phi \supseteq \Delta_\Gamma$, for all deductively closed sets of sentences Φ.*

Proof: The only nontrivial part is the deductive closure of Δ_Γ; but this follows easily from the axiomatization of BQC. □

The theory Δ_Γ need not be saturated. Also note that $\Gamma \prec \Delta \subseteq \Phi$ does not imply that $\Gamma \prec \Phi$. For example, set $\Gamma = \Delta = \{p \to q\}$, and $\Phi = \{p \to q, p\}$, for atomic p and q.

Proposition 3.3 *Let $\Gamma \subseteq \Delta$ be such that $\Gamma \prec \Delta \nvdash A$. Then there is a deductively closed saturated $\Delta' \supseteq \Delta$ such that $\Gamma \prec \Delta' \nvdash A$.*

Proof: Let $(A_i)_i$ be a countable enumeration of all sentences of the language of $\Delta = \Delta^0$, and E be a countable set of new constant symbols. We may assume $E = (e_i)_i$ for some enumeration $i \mapsto e_i$. Form an ascending chain of sets $(\Delta_i)_i$ as follows: $\Delta_0 = \Delta^0$. Given Δ_i, construct Δ_{i+1} as follows:

1. A_i is of the form $B \vee C$ and $\Delta_i \vdash B \vee C$. If $(\Delta_i \cup \{B\})_\Gamma \nvdash A$, set $\Delta_{i+1} = (\Delta_i \cup \{B\})_\Gamma$; otherwise set $\Delta_{i+1} = (\Delta_i \cup \{C\})_\Gamma$.
2. A_i is of the form $\exists y B$ and $\Delta_i \vdash \exists y B$. Set $\Delta_{i+1} = (\Delta_i \cup \{B^y_e\})_\Gamma$, where i is the smallest index such that $e = e_i$ is not used in Δ_i.
3. Otherwise, set $\Delta_{i+1} = \Delta_i$.

Set $\Delta^1 = \cup_i \Delta_i$. Repeat the same process to construct Δ^2 from Δ^1, and so on. Then $\Delta' = \cup_i \Delta^i$ is a deductively closed saturated theory such that $\Gamma \prec \Delta'$. It remains to show that $\Delta' \nvdash A$. It suffices to check the two steps involving disjunction and existential quantification to complete the usual inductive proof. If $\Delta_i \vdash B \vee C$, and both $(\Delta_i \cup \{B\})_\Gamma \vdash A$ and $(\Delta_i \cup \{C\})_\Gamma \vdash A$, then $\Gamma \vdash \forall \mathbf{x}(D^{\mathbf{c}}_{\mathbf{x}} \wedge (B^{\mathbf{c}}_{\mathbf{x}} \vee C^{\mathbf{c}}_{\mathbf{x}}) \to A^{\mathbf{c}}_{\mathbf{x}})$ with $\Delta_i \vdash D$ for some D, and $\mathbf{c}$ includes all constant symbols not in the language of Γ; so $\Delta_i \vdash A$. Similarly, if $\Delta_i \vdash \exists y B$ and $(\Delta_i \cup \{B^y_e\})_\Gamma \vdash A$, then $\Gamma \vdash \forall \mathbf{x}(D^{\mathbf{c}}_{\mathbf{x}} \wedge (\exists y B)^{\mathbf{c}}_{\mathbf{x}} \to A^{\mathbf{c}}_{\mathbf{x}})$ with $\Delta_i \vdash D$ for some D and some $\mathbf{c}$; so $\Delta_i \vdash A$. So if $\Delta_i \nvdash A$, then $\Delta_{i+1} \nvdash A$. □

We *also* need the following proposition.

Proposition 3.4 $\varnothing \prec \Gamma$.

Proof: It follows immediately from the axiomatization of BQC that if a sentence $\forall \mathbf{x}(A \to B)$ is derivable, then so is the sequent $A^{\mathbf{x}}_{\mathbf{c}} \Longrightarrow B^{\mathbf{x}}_{\mathbf{c}}$. □

So if $\Gamma \nvdash A$, then some saturated $\Gamma' \supseteq \Gamma$ exists with $\Gamma' \nvdash A$.

Theorem 3.5 (Completeness) *Let $\Gamma \cup \{A\}$ be a countable set of sentences. Then $\Gamma \models A$ implies $\Gamma \vdash A$.*

Proof: Suppose $\Gamma \nvdash A$. We construct a Kripke model as follows. As set of worlds W we choose the collection of deductively closed saturated countable sets of sentences containing Γ. We set $\Delta \prec \Delta'$ as specified above for deductively closed sets. The construction of the domain $D\Delta$ above a node Δ takes only two steps. Start with the countable set of constant symbols E of the language of Δ. Set $d \sim e$ whenever $(d = e) \in \Delta$. It is immediate from the axiomatization of BQC that $\sim$ is a congruence on E. Set $D\Delta$ equal to the collection of equivalence classes $D\Delta = E_\sim$. If $\Delta \prec \Delta'$, then $\Delta \subseteq \Delta'$, so the inclusion function $E \subseteq E'$ respects the congruences. For atomic sentences A, set $\Delta \Vdash A$ if and only if $A \in \Delta$. An easy proof by induction on the complexity of sentences shows that $\Delta \Vdash B$ if and only if $B \in \Delta$, for all worlds Δ and B in the language of Δ. The only two nontrivial steps involve the quantifiers. For example, $\exists x B \in \Delta$, if and only if (by saturatedness) $B^x_c \in \Delta$ for some c with

$c_\sim \in D\Delta$, if and only if (induction step) $\Delta \Vdash B^x_c$ for some c with $c_\sim \in D\Delta$, if and only if $\Delta \Vdash \exists x B$. As to the universal quantifier case, suppose $\forall \mathbf{x}(B \to C) \in \Delta$. If $\Delta \prec \Delta' \Vdash B^{\mathbf{x}}_{\mathbf{c}}$ then, by induction, $B^{\mathbf{x}}_{\mathbf{c}} \in \Delta'$ so, by definition of $\prec$ and induction, $\Delta' \Vdash C^{\mathbf{x}}_{\mathbf{c}}$. So $\Delta \Vdash \forall \mathbf{x}(B \to C)$. Conversely, suppose $\forall \mathbf{x}(B \to C) \notin \Delta$. Let $\mathbf{c}$ be a sequence of new constant symbols, and set $\Delta_0 = (\Delta \cup \{B^{\mathbf{x}}_{\mathbf{c}}\})_\Delta$. By Proposition 3.2, $\Delta \prec \Delta_0$. Suppose $\Delta_0 \vdash C^{\mathbf{x}}_{\mathbf{c}}$. Then $\Delta \vdash \forall \mathbf{x}(D \wedge B \to C)$ for some D such that $\Delta \vdash D^{\mathbf{x}}_{\mathbf{c}}$; so also $\Delta \vdash \forall \mathbf{x}(\top \to D)$ and $\Delta \vdash \forall \mathbf{x}(B \to C)$, contradiction. So $\Delta_0 \nvdash C^{\mathbf{x}}_{\mathbf{c}}$. By Proposition 3.3, there is a node $\Delta' \succ \Delta$ with $\Delta' \nvdash C^{\mathbf{x}}_{\mathbf{c}}$. By induction, $\Delta' \Vdash B^{\mathbf{x}}_{\mathbf{c}}$ and $\Delta' \nVdash C^{\mathbf{x}}_{\mathbf{c}}$. So $\Delta \nVdash \forall \mathbf{x}(B \to C)$. All nodes contain Γ, so the model satisfies Γ. But by Propositions 3.3 and 3.4, some node doesn't contain A. So the model doesn't satisfy A. And thus $\Gamma \not\models A$. □

4 Terms and theories What about theories over BQC that contain rules and sequents as part of their axiomatization? One example of such a theory is Basic Arithmetic (BA), the basic logic equivalent of intuitionistic Heyting Arithmetic (HA) and classical Peano Arithmetic (PA), see Section 6. It turns out that these more general theories must be functional and well-formed. Both notions are discussed in this section. In Section 5 we will show that these two properties are both necessary and sufficient in proving the stronger completeness theorem. As the concepts are rather sensitive to detail, we present the theory in extra detail. There is another reason to be precise. It is not unusual to present completeness theorems for more well-known systems in a slightly informal way. The basic idea shines through, though often this clarity comes at the expense of leaving out significant details that are considered obvious. The omissions don't hurt because we all know the result to be correct from other proofs of this particular completeness theorem presented elsewhere. Unfortunately, for BQC we don't (yet) have this stable situation. Therefore we feel compelled to be more pedantic than we otherwise would have been.

The standard examples of theories are IQC, FQC, and CQC: Intuitionistic Predicate Calculus (IQC) is the extension of BQC axiomatizable by all substitution instances of the Rule of Modus Ponens

$$\frac{A \Longrightarrow B \to C}{A \wedge B \Longrightarrow C}\ .$$

This trivially implies that IQC also entails $A \Longrightarrow \forall \mathbf{x}(B \to C) \vdash A \wedge B \Longrightarrow C$. Formal Predicate Calculus (FQC) is the extension of BQC axiomatizable by all substitution instances of Löb's Rule

$$\frac{A \wedge (\top \to B) \Longrightarrow B}{A \Longrightarrow B}\ .$$

Classical Predicate Calculus (CQC) is the extension of IQC axiomatized by adding all substitution instances of Excluded Middle

$$A \vee \neg A.$$

The definitions of IQC, FQC, and CQC are close analogons of the definitions of IPC, FPC, and CPC, as described in [1]. The proof of the following proposition is essentially identical to the one in [1].

Proposition 4.1 *IQC is axiomatizable by the schema*

$$\top \to A \Longrightarrow A.$$

FQC is axiomatizable by the schema (Löb's Axiom)

$$(\top \to A) \to A \Longrightarrow \top \to A.$$

So IQC and FQC are sequent theories, and so is CQC.

The universal quantification rules and axiom schemas allow for the following simple conservativity property.

Proposition 4.2 *BQC is conservative over its geometric fragment, that is, if* $\Gamma \cup \{\gamma\}$ *is a set of geometric sequents such that* $\Gamma \vdash \gamma$*, then there is a derivation tree of* γ *from* Γ *that does not use the universal quantifier symbol.*

Proof: For each rule and axiom schema of BQC, if we replace the universal quantification subformulas by $\top$, then the resulting rule or axiom schema is derivable from BQC by a different instance of the same rule or axiom, or is an instance of the schema $A \Longrightarrow \top$. So given a derivation tree of γ from Γ, replace all occurrences of universal quantifier subformulas by $\top$, and replace the resulting new rules and axioms by the simpler derivations in BQC. The result is a geometric logic derivation tree of γ from Γ. □

Variable-free formula contexts are defined as in [1]: add a new 0-ary predicate symbol p to the language. Let $D[p]$ be a formula over the extended language, and let A be a formula over the original language. Then $D[A]$ is constructed by replacing each occurrence of p by A. Similarly, multiple, say double, simultaneous formula substitutions are performed by adding new 0-ary predicate symbols p and q to the language. If A and B are formulas over the original language, then $D[A,B]$ is formed from $D[p,q]$ by replacing all occurrences of p by A, and all occurrences of q by B. This translation process is called *simple substitution*. Later on we will introduce the concept of proper substitution. Proper substitution allows for more refined versions of substitution, but it is a significantly more technical tool. We wish to postpone these complications in exchange for simplicity.

Proposition 4.3 (Formula substitution) *BQC is closed under the substitution rule*

$$\frac{A \wedge B \Longrightarrow C \quad A \wedge C \Longrightarrow B}{A \wedge D[B] \Longrightarrow D[C]}\ ,$$

where p *does not occur within range of a quantifier of* $D[p]$ *over a variable that is free in both* A *and* $B \wedge C$*.*

Proof: We complete the proof by induction on the complexity of $D[p]$. Let Γ be the theory axiomatized by $A \wedge B \Longrightarrow C$ and $A \wedge C \Longrightarrow B$. The cases where $D[p]$ is atomic or doesn't contain p are trivial. If $D[p]$ equals $E[p] \vee F[p]$ then, by induction, we immediately derive $\Gamma \vdash A \wedge E[B] \Longrightarrow E[C] \vee F[C]$ and $\Gamma \vdash A \wedge F[B] \Longrightarrow E[C] \vee F[C]$. Simple propositional logic then gives us $\Gamma \vdash A \wedge (E[B] \vee F[B]) \Longrightarrow E[C] \vee F[C]$. Suppose $D[p]$ equals $E[p] \wedge F[p]$. By induction, $\Gamma \vdash A \wedge E[B] \Longrightarrow$

$E[C]$ and $\Gamma \vdash A \wedge F[B] \Longrightarrow F[C]$. So $\Gamma \vdash A \wedge E[B] \wedge F[B] \Longrightarrow E[C] \wedge F[C]$. Suppose $D[p]$ equals $\exists x E[p]$. By induction $\Gamma \vdash A \wedge E[B] \Longrightarrow E[C]$. BQC entails the schema $G \Longrightarrow \exists y G$, so we have $\Gamma \vdash A \wedge E[B] \Longrightarrow \exists x E[C]$. Now x is not free in A, or not free in $B \wedge C$. Suppose x is not free in A. Then $\Gamma \vdash \exists x(A \wedge E[B]) \Longrightarrow \exists x E[C]$, thus also $\Gamma \vdash A \wedge \exists x E[B] \Longrightarrow \exists x E[C]$. Otherwise, x is not free in $B \wedge C$. Replace all free occurrences of x in A by a new variable x', resulting in a new formula A'. Let Γ' be axiomatized by $A' \wedge B \Longrightarrow C$ plus $A' \wedge C \Longrightarrow B$. Then, as before, $\Gamma' \vdash A' \wedge \exists x E[B] \Longrightarrow \exists x E[C]$. So, by variable substitution, there is also a proof for $\Gamma \vdash A \wedge \exists x E[B] \Longrightarrow \exists x E[C]$. Finally, suppose $D[p]$ equals $\forall \mathbf{x}(E[p] \to F[p])$. By induction we have $\Gamma \vdash A \wedge E[C] \Longrightarrow E[B]$ and $\Gamma \vdash A \wedge F[B] \Longrightarrow F[C]$. Now x is not free in A, or not free in $B \wedge C$, for all x occurring in $\mathbf{x}$. If no variable in $\mathbf{x}$ is free in A, then $\Gamma \vdash A \Longrightarrow \forall \mathbf{x}(E[C] \to E[B])$ and $\Gamma \vdash A \Longrightarrow \forall \mathbf{x}(F[B] \to F[C])$. Thus $\Gamma \vdash A \wedge \forall \mathbf{x}(E[B] \to F[B]) \Longrightarrow \forall \mathbf{x}(E[C] \to F[C])$. Otherwise, some variables in $\mathbf{x}$ are free in A, hence not free in $B \wedge C$. Replace all free occurrences of these variables of $\mathbf{x}$ in A by new variables $\mathbf{x}'$, resulting in a new formula A'. Let Γ' be axiomatized by $A' \wedge B \Longrightarrow C$ plus $A' \wedge C \Longrightarrow B$. Then, as before, $\Gamma' \vdash A' \wedge \forall \mathbf{x}(E[B] \to F[B]) \Longrightarrow \forall \mathbf{x}(E[C] \to F[C])$. So, by variable substitution, there is also a proof for $\Gamma \vdash A \wedge \forall \mathbf{x}(E[B] \to F[B]) \Longrightarrow \forall \mathbf{x}(E[C] \to F[C])$. □

The following proposition makes precise what we mean by renaming bound variables.

Proposition 4.4 (Renaming of bound variables) *Let A and B be formulas in which the variables in $\mathbf{x}$ and $\mathbf{y}$ do not occur freely, and where no variable in $\mathbf{x}$ or $\mathbf{y}$ becomes bound during substitution in $A^{\mathbf{z}}_{\mathbf{x}}$, $B^{\mathbf{z}}_{\mathbf{x}}$, $A^{\mathbf{z}}_{\mathbf{y}}$, or $B^{\mathbf{z}}_{\mathbf{y}}$. Then BQC proves*

$$D[\forall \mathbf{x}(A^{\mathbf{z}}_{\mathbf{x}} \to B^{\mathbf{z}}_{\mathbf{x}})] \Longleftrightarrow D[\forall \mathbf{y}(A^{\mathbf{z}}_{\mathbf{y}} \to B^{\mathbf{z}}_{\mathbf{y}})]$$

for all contexts $D[p]$. Let C be a formula in which the variables x and y do not occur freely, and where neither x nor y becomes bound during substitution in C^z_x or C^z_y. Then BQC proves

$$D[\exists x C^z_x] \Longleftrightarrow D[\exists y C^z_y]$$

for all contexts $D[p]$.

Proof: Let A and B be formulas satisfying the required conditions. Then BQC entails $\forall \mathbf{x}(A^{\mathbf{z}}_{\mathbf{x}} \to B^{\mathbf{z}}_{\mathbf{x}}) \Longleftrightarrow \forall \mathbf{xy}(A^{\mathbf{z}}_{\mathbf{x}} \to B^{\mathbf{z}}_{\mathbf{x}}) \Longleftrightarrow \forall \mathbf{xy}(A^{\mathbf{z}}_{\mathbf{y}} \to B^{\mathbf{z}}_{\mathbf{y}}) \Longleftrightarrow \forall \mathbf{y}(A^{\mathbf{z}}_{\mathbf{y}} \to B^{\mathbf{z}}_{\mathbf{y}})$. Apply formula substitution. Let C be a formula satisfying the required conditions. Then BQC entails $C^z_y \Longrightarrow \exists y C^z_y$ so, by variable substitution, also $C^z_x \Longrightarrow \exists y C^z_y$, and thus $\exists x C^z_x \Longrightarrow \exists y C^z_y$. By symmetry we also have $\exists y C^z_y \Longrightarrow \exists x C^z_x$. Apply formula substitution. □

So renaming bound variables doesn't change a formula in an essential way. Moreover, if $\mathbf{x}$ and $\mathbf{x}'$ are two sequences consisting of the same variables, but maybe in different orders and with different multiplicities of occurrence, then BQC obviously proves $\forall \mathbf{x}(A \to B) \Longleftrightarrow \forall \mathbf{x}'(A \to B)$. Combined with repeated application of Proposition 4.4 this allows us to make all quantifier variables different from one another and from the free variables.

As with formulas we write $\gamma^{\mathbf{x}}_{\mathbf{t}}$ for substitution of terms of $\mathbf{t}$ for the variables of $\mathbf{x}$ in all their free occurrences in the sequent γ. Given a language $\mathcal{L}$ and a set or sequence of new constant symbols $\mathbf{c}$, we write $\mathcal{L}[\mathbf{c}]$ for the extension language.

Proposition 4.5 (Generalization) *Let Γ be a set of sequents and rules over a language $\mathcal{L}$, and γ be a sequent over $\mathcal{L}$. Let $\mathbf{c}$ be a sequence of new constant symbols and $\mathbf{x}$ a matching sequence of variables. Then $\Gamma \vdash \gamma$ over $\mathcal{L}$, if and only if $\Gamma \vdash \gamma^{\mathbf{x}}_{\mathbf{c}}$ over $\mathcal{L}[\mathbf{c}]$.*

Proof: From left to right, apply variable substitution: BQC proves $\gamma \vdash \gamma^{\mathbf{x}}_{\mathbf{c}}$. We complete the proof of the converse by induction on the complexity of derivations. If $\gamma^{\mathbf{x}}_{\mathbf{c}}$ is an axiom sequent, then so is γ. So it remains to check the rules. If the last rule applied is from Γ, then $\gamma^{\mathbf{x}}_{\mathbf{c}}$ equals γ. Only the rules of BQC are left. The quantifier-free ones are easy. Suppose that $B^{\mathbf{y}}_{\mathbf{t}} \Longrightarrow C^{\mathbf{y}}_{\mathbf{t}}$ follows from $B \Longrightarrow C$ by variable substitution. Replace all occurrences of constant symbols from $\mathbf{c}$ by new variables from $\mathbf{z}$. By induction hypothesis, $\Gamma \vdash B^{\mathbf{c}}_{\mathbf{z}} \Longrightarrow C^{\mathbf{c}}_{\mathbf{z}}$, and two successive applications of variable substitution give us $(B^{\mathbf{c}}_{\mathbf{z}})^{\mathbf{y}}_{\mathbf{t}} \Longrightarrow (C^{\mathbf{c}}_{\mathbf{z}})^{\mathbf{y}}_{\mathbf{t}}$, and then $(B^{\mathbf{y}}_{\mathbf{t}})^{\mathbf{c}}_{\mathbf{z}} \Longrightarrow (C^{\mathbf{y}}_{\mathbf{t}})^{\mathbf{c}}_{\mathbf{z}}$. So $\Gamma \vdash (B^{\mathbf{y}}_{\mathbf{t}})^{\mathbf{c}}_{\mathbf{x}} \Longrightarrow (C^{\mathbf{y}}_{\mathbf{t}})^{\mathbf{c}}_{\mathbf{x}}$. Suppose that C has no free occurrence of x, and $\exists x D \Longrightarrow C$ follows from $D \Longrightarrow C$. By induction hypothesis, $\Gamma \vdash (D \Longrightarrow C)^{\mathbf{c}}_{\mathbf{z}}$, where $\mathbf{z}$ is a sequence of new variables. So $\Gamma \vdash (\exists x D \Longrightarrow C)^{\mathbf{c}}_{\mathbf{z}}$, and thus $\Gamma \vdash (\exists x D \Longrightarrow C)^{\mathbf{c}}_{\mathbf{x}}$. Suppose $B \Longrightarrow C$ follows from $\exists x B \Longrightarrow C$. Then, by induction hypothesis, $\Gamma \vdash (\exists x B \Longrightarrow C)^{\mathbf{c}}_{\mathbf{z}}$, with $\mathbf{z}$ all new variables. So $\Gamma \vdash (B \Longrightarrow C)^{\mathbf{c}}_{\mathbf{z}}$, and thus $\Gamma \vdash (B \Longrightarrow C)^{\mathbf{c}}_{\mathbf{x}}$. Suppose that no variable in $\mathbf{y}$ is free in A, and $A \Longrightarrow \forall \mathbf{y}(B \to C)$ follows from $A \wedge B \Longrightarrow C$. By induction hypothesis, $\Gamma \vdash (A \wedge B \Longrightarrow C)^{\mathbf{c}}_{\mathbf{z}}$, where $\mathbf{z}$ is a sequence of new variables. So $\Gamma \vdash (A \Longrightarrow \forall \mathbf{y}(B \to C))^{\mathbf{c}}_{\mathbf{z}}$. By variable substitution, as the constant symbols of $\mathbf{c}$ only occur in places where the corresponding variables of $\mathbf{x}$ are free, $\Gamma \vdash (A \Longrightarrow \forall \mathbf{y}(B \to C))^{\mathbf{c}}_{\mathbf{x}}$. □

BQC proves $B \Longrightarrow A \vdash \exists x B \Longrightarrow A$ if x is not free in A, but in general not $B^x_c \Longrightarrow A \vdash \exists x B \Longrightarrow A$. So, even if no variable in $\mathbf{x}$ is free in γ, Proposition 4.5 cannot be broadened to place constants on the left of the turnstile.

When we extend the language, say language $\mathcal{L}$, by adding new predicates or function symbols, then BQC changes along. These changes are harmless in the sense that the extension is conservative over the original language.

Proposition 4.6 *Let $\mathcal{L} \subseteq \mathcal{M}$ be languages, and $\Gamma \cup \{\gamma\}$ be a set of sequents over $\mathcal{L}$. Then $\Gamma \vdash \gamma$ over $\mathcal{L}$, if and only if $\Gamma \vdash \gamma$ over $\mathcal{M}$.*

Proof: Given a (finite) proof tree over the extended language whose assumptions and conclusion are in the original language, replace all new predicates by $\top$, and all new function symbols by fresh variables. The new tree is a proof of the same, but over the original language. □

Proposition 4.6 can be strengthened to theories with schemas as follows. Formula contexts are defined by adding new predicate symbols to the language. Similarly, sequent and rule contexts are defined by employing new predicate letters in the formulas that are part of these sequents or rules. *Proper substitution of formulas*, a refinement of simple substitution, is defined as follows: for each n-ary new predicate symbol p

we can choose a formula A over the original language, whose free variables that already occur in the context (formula, sequent, or rule) are all from among a list $\mathbf{x}$ of n variables. Each atomic expression $p\mathbf{t}$ can then be replaced by $A^{\mathbf{x}}_{\mathbf{t}}$ subject to the condition that no free variable in the sequence $\mathbf{t}$ becomes bound in $A^{\mathbf{x}}_{\mathbf{t}}$ after substitution. For example, variable substitution could be redefined as all proper substitutions into the schemas

$$\frac{p\mathbf{x} \Longrightarrow q\mathbf{x}}{p\mathbf{t} \Longrightarrow q\mathbf{t}} ,$$

where p and q are of equal arity and range over countable sequences of new predicate symbols $p_1, p_2, p_3, \ldots$ and $q_1, q_2, q_3, \ldots$ respectively, where p_n and q_n are n-ary, and the sequences of variables $\mathbf{x}$ and terms $\mathbf{t}$ are such that the atoms are well-formed. The notion of context can be further generalized by also adding new function symbols. *Proper substitution of terms* into contexts is analogous to proper formula substitution: for each n-ary new function symbol f we can choose a term u over the original language, whose free variables that already occur in the context are all from among a list $\mathbf{x}$ of n free variables; each term expression $f(\mathbf{t})$ can then be replaced by $u^{\mathbf{x}}_{\mathbf{t}}$.

A *schematic axiomatization* for a theory Γ over a language $\mathcal{L}$ consists of a set $\mathcal{S}$ of sequents and rules over a larger language $\mathcal{L}[P, F]$ satisfying the following properties:

1. $\mathcal{L}[P, F]$ is constructed from $\mathcal{L}$ by augmenting it with a set of new predicate symbols P and a set of new function symbols F;
2. each rule or sequent that is obtained from $\mathcal{S}$ by a proper substitution of formulas of $\mathcal{L}$ for all predicate letters from P and of terms of $\mathcal{L}$ for all function symbols from F, is in Γ; and
3. Γ is the smallest theory satisfying these properties.

Each rule or sequent over $\mathcal{L}$ that is derivable from $\mathcal{S}$ is in Γ; and each theory is schematically axiomatized by itself over $\mathcal{L}$. The following generalizes Proposition 4.6.

Proposition 4.7 *Let $\mathcal{L} \subseteq \mathcal{M}$ be languages, and let $\mathcal{S} \subseteq \mathcal{L}[P, F]$, with $\mathcal{L}[P, F] \cap \mathcal{M} = \mathcal{L}$, be a schematic axiomatization of theories Γ over $\mathcal{L}$ and Δ over $\mathcal{M}$. Let γ be a sequent over $\mathcal{L}$. Then $\Gamma \vdash \gamma$ over $\mathcal{L}$, if and only if $\Delta \vdash \gamma$ over $\mathcal{M}$.*

Proof: As for Proposition 4.6. □

In general the theory Γ need not be contained in the theory generated by $\mathcal{S}$, for the theory generated by $\mathcal{S}$ isn't closed under proper substitution. Rather, let Δ be the theory over $\mathcal{L}[P, F]$, axiomatized by $\mathcal{S}$ and all proper substitutions of formulas and terms from $\mathcal{L}$ into the rules and sequents of $\mathcal{S}$. Then $\Gamma = \mathcal{L} \cap \Delta$, if and only if $\mathcal{S}$ is a schematic axiomatization of Γ. If so, then Δ is also a schematic axiomatization of Γ.

A basic class of schematic axiomatizations is formed by the minimal ones:

Proposition 4.8 *Let $\mathcal{L} \subseteq \mathcal{M}$ be languages, let Γ be a theory over $\mathcal{L}$, and let Δ be the theory over $\mathcal{M}$ axiomatized by Γ. Then Δ schematically axiomatizes Γ. In particular, BQC over a fixed language $\mathcal{L}$ is schematically axiomatized by the theory BQC of any extension language, and BQC proves*

$$x = y \wedge A \Longrightarrow A^x_y$$

for all formulas A where y does not become bound after substitution.

Proof: The general case immediately follows from the case for BQC. But all rule and sequent axiom schemas of BQC are closed under proper substitution, except the equality schema $x = y \wedge A \Longrightarrow A^x_y$. So it suffices to prove the last part of this proposition. We leave it as an easy exercise to show that BQC entails $x = y \Longrightarrow y = x$. We complete the proof by induction on the complexity of formulas. The equality schema holds for atoms. It is an easy exercise to check all induction cases except the quantifier ones. Let x and y both be different from z, such that $x = y \wedge A \Longrightarrow A^x_y$. Since BQC proves $A^x_y \Longrightarrow (\exists z A)^x_y$, we have $x = y \wedge A \Longrightarrow (\exists z A)^x_y$, and thus $x = y \wedge \exists z A \Longrightarrow \exists z(x = y \wedge A) \Longrightarrow (\exists z A)^x_y$. Let x and y both be different from all variables in the sequence $\mathbf{z}$, such that $x = y \wedge A^x_y \Longrightarrow A$ and $x = y \wedge B \Longrightarrow B^x_y$. So we have $x = y \Longrightarrow \forall \mathbf{z}(A^x_y \to A) \wedge \forall \mathbf{z}(B \to B^x_y)$. And thus $x = y \wedge \forall \mathbf{z}(A \to B) \Longrightarrow (\forall \mathbf{z}(A \to B))^x_y$. □

Let Γ be a theory over $\mathcal{L}$, and let C be a set of new constant symbols. Write $\Gamma[C]$ for the theory over $\mathcal{L}[C]$ axiomatized by Γ. Let Δ be another theory over $\mathcal{L}[C]$ that schematically axiomatizes Γ. Let γ be a sequent over $\mathcal{L}$, and $\mathbf{c}$ be a sequence of new constant symbols, such that $\Delta \vdash \gamma^{\mathbf{x}}_{\mathbf{c}}$. Then $\Gamma[C] \supseteq \Gamma \vdash \gamma$, so $\Gamma[C] \vdash \gamma^{\mathbf{x}}_{\mathbf{c}}$. So $\Gamma[C]$ is the maximal sequent theory over Γ in $\mathcal{L}[C]$ that schematically axiomatizes Γ, hence it is the unique one containing Γ. Uniqueness usually fails when we admit new predicate symbols.

4.1 Functional theories A crucial property of theories is functional completeness, see below. It is a natural property that is vital in Kripke model theory. We describe the theories for which all sequent theory extensions satisfy this.

A set of sequents and rules $\Gamma \subseteq \mathcal{L}$ is *functional* over $\mathcal{L}$ if for all rules

$$\frac{A_1 \Longrightarrow B_1 \quad \dots \quad A_n \Longrightarrow B_n}{A_0 \Longrightarrow B_0} \in \Gamma$$

and sentences $A \in \mathcal{L}$,

$$\Gamma \cup \{A \wedge A_1 \Longrightarrow B_1, \dots, A \wedge A_n \Longrightarrow B_n\} \vdash A \wedge A_0 \Longrightarrow B_0 .$$

A theory Γ over $\mathcal{L}$ is *functional* if for all sequences of formulas $A_0, B_0, A_1, B_1, \dots, A_n, B_n \in \mathcal{L}$ and sentences $A \in \mathcal{L}$, if

$$\Gamma \cup \{A_1 \Longrightarrow B_1, \dots, A_n \Longrightarrow B_n\} \vdash A_0 \Longrightarrow B_0 ,$$

then

$$\Gamma \cup \{A \wedge A_1 \Longrightarrow B_1, \dots, A \wedge A_n \Longrightarrow B_n\} \vdash A \wedge A_0 \Longrightarrow B_0 .$$

A theory Γ has a *functional axiomatization* over $\mathcal{L}$, if it is axiomatizable by a set which is functional over $\mathcal{L}$.

Proposition 4.9 *A theory over $\mathcal{L}$ is functional if and only if it has a functional axiomatization over $\mathcal{L}$.*

Proof: A functional theory is axiomatized by itself. Conversely, suppose a theory is axiomatized by a functional set Γ over $\mathcal{L}$. Suppose formulas A_0, B_0, A_1, $B_1, \ldots, A_n$, $B_n \in \mathcal{L}$ and sentence $A \in \mathcal{L}$ are such that

$$\Gamma \cup \{A_1 \Longrightarrow B_1, \ldots, A_n \Longrightarrow B_n\} \vdash A_0 \Longrightarrow B_0\,,$$

in order to show that

$$\Gamma \cup \{A \wedge A_1 \Longrightarrow B_1, \ldots, A \wedge A_n \Longrightarrow B_n\} \vdash A \wedge A_0 \Longrightarrow B_0\,.$$

We complete the proof by induction on the complexity of derivations. The cases when $\Gamma \vdash A_0 \Longrightarrow B_0$, or $A_0 \Longrightarrow B_0$ equals some $A_i \Longrightarrow B_i$, $i > 0$, are immediate. If the last rule in the derivation of $A_0 \Longrightarrow B_0$ involves a rule of Γ, then the result immediately follows from the functionality of Γ. So it remains to check the induction step for rules of BQC. But that only requires that we verify the functionality of the BQC rules. For example, if

$$\frac{B \Longrightarrow C}{B^{\mathbf{x}}_{\mathbf{t}} \Longrightarrow C^{\mathbf{x}}_{\mathbf{t}}}$$

is an instance of variable substitution, and A is a sentence, then

$$A \wedge B \Longrightarrow C \vdash A \wedge B^{\mathbf{x}}_{\mathbf{t}} \Longrightarrow C^{\mathbf{x}}_{\mathbf{t}}.$$

The remaining BQC rules are just as easy. □

Corollary 4.10 *BQC is functional. Sequent theory extensions of functional theories are functional.*

For all theories Γ and formulas A, B, and C, if $\Gamma \vdash A \wedge B \Longrightarrow C$, then $\Gamma \cup \{A\} \vdash B \Longrightarrow C$. A theory $\Gamma \subseteq \mathcal{L}$ is *functionally complete* over $\mathcal{L}$ if for all formulas $B, C \in \mathcal{L}$ and sentences $A \in \mathcal{L}$, if $\Gamma \cup \{A\} \vdash B \Longrightarrow C$, then $\Gamma \vdash A \wedge B \Longrightarrow C$.

Proposition 4.11 (Functional completeness) *A theory is functional over $\mathcal{L}$ if and only if all sequent theory extensions are functionally complete over $\mathcal{L}$.*

Proof: Let Δ be a sequent theory extension of a functional theory Γ, and formulas $B, C \in \mathcal{L}$ and sentence $A \in \mathcal{L}$ be such that $\Delta \cup \{A\} \vdash B \Longrightarrow C$. There are sequents $A_i \Longrightarrow B_i \in \Delta$, $1 \leq i \leq n$, such that

$$\Gamma \cup \{A_1 \Longrightarrow B_1, \ldots, A_n \Longrightarrow B_n, A\} \vdash B \Longrightarrow C.$$

By functionality,

$$\Gamma \cup \{A \wedge A_1 \Longrightarrow B_1, \ldots, A \wedge A_n \Longrightarrow B_n, A \Longrightarrow A\} \vdash A \wedge B \Longrightarrow C.$$

So $\Delta \vdash A \wedge B \Longrightarrow C$. Conversely, suppose that all sequent theory extensions of Γ are functionally complete. Let $A_i \Longrightarrow B_i$, $0 \leq i \leq n$, be sequents such that

$$\Gamma \cup \{A_1 \Longrightarrow B_1, \ldots, A_n \Longrightarrow B_n\} \vdash A_0 \Longrightarrow B_0.$$

So for sentences $A \in \mathcal{L}$,

$$\Gamma \cup \{A \wedge A_1 \Longrightarrow B_1, \ldots, A \wedge A_n \Longrightarrow B_n, A\} \vdash A_0 \Longrightarrow B_0.$$

Apply functional completeness of $\Gamma \cup \{A \wedge A_1 \Longrightarrow B_1, \ldots, A \wedge A_n \Longrightarrow B_n\}$. □

Functionality of a set of sequents and rules is essentially language dependent. This is particularly relevant in the construction of Kripke models in Section 5. We are mainly interested in extensions of languages by new constant symbols. Let Γ be a set of sequents and rules over $\mathcal{L}$, and C be a set of new constant symbols. Define $\Gamma\langle C\rangle$ to be the set Γ extended by all rules of the form

$$A \times R = \frac{A \wedge A_1 \Longrightarrow B_1 \quad \ldots \quad A \wedge A_n \Longrightarrow B_n}{A \wedge A_0 \Longrightarrow B_0} \ ,$$

where A is a sentence of $\mathcal{L}[C]$ and

$$R = \frac{A_1 \Longrightarrow B_1 \quad \ldots \quad A_n \Longrightarrow B_n}{A_0 \Longrightarrow B_0} \in \Gamma \ .$$

It is immediate from the definitions that $\Gamma\langle C\rangle$ is a functional set over $\mathcal{L}[C]$. The theory axiomatized by $\Gamma\langle C\rangle$ is also written $\Gamma\langle C\rangle$. It is the smallest functional theory over $\mathcal{L}[C]$ containing Γ.

Proposition 4.12 (Functional generalization) *Let Γ be a functional set of sequents and rules over a language $\mathcal{L}$, and $\mathbf{c}$ be a sequence of elements from a set C of new constant symbols, and $\mathbf{x}$ a matching sequence of variables. Let γ be a sequent over $\mathcal{L}$. Then $\Gamma \vdash \gamma$ over $\mathcal{L}$, if and only if $\Gamma\langle C\rangle \vdash \gamma^{\mathbf{x}}_{\mathbf{c}}$ over $\mathcal{L}[C]$.*

Proof: From left to right is immediate by variable substitution. Conversely, suppose $\Gamma\langle C\rangle \vdash \gamma^{\mathbf{x}}_{\mathbf{c}}$. We complete the proof by induction on the complexity of derivations. All induction steps are identical to the ones in the proof of Generalization Proposition 4.5, except for the new rules of $\Gamma\langle C\rangle \setminus \Gamma$. Suppose the last step in the proof of $\gamma^{\mathbf{x}}_{\mathbf{c}}$ is the rule $A \times R$, where

$$R = \frac{A_1 \Longrightarrow B_1 \quad \ldots \quad A_n \Longrightarrow B_n}{A_0 \Longrightarrow B_0} \in \Gamma \ ,$$

$A \in \mathcal{L}[C]$ a sentence, and $\gamma^{\mathbf{x}}_{\mathbf{c}}$ equals $A \wedge A_0 \Longrightarrow B_0$. Let $\mathbf{y}$ be a sequence of proper length of new variables. By induction, $\Gamma \vdash A^{\mathbf{c}}_{\mathbf{y}} \wedge A_i \Longrightarrow B_i$, for all $i > 0$. So $\Gamma \vdash (\exists \mathbf{y} A^{\mathbf{c}}_{\mathbf{y}}) \wedge A_i \Longrightarrow B_i$, for all $i > 0$. Γ is functional, so $\Gamma \vdash (\exists \mathbf{y} A^{\mathbf{c}}_{\mathbf{y}}) \wedge A_0 \Longrightarrow B_0$. So $\Gamma \vdash A^{\mathbf{c}}_{\mathbf{y}} \wedge A_0 \Longrightarrow B_0$, and thus $\Gamma \vdash A^{\mathbf{c}}_{\mathbf{x}} \wedge A_0 \Longrightarrow B_0$. □

4.2 Well-formed theories A set $\Gamma \subseteq \mathcal{L}$ is *well-formed* if for all sequences of sentences $\forall \mathbf{x}(A_0 \to B_0), \forall \mathbf{x}(A_1 \to B_1), \ldots, \forall \mathbf{x}(A_n \to B_n)$ and formulas A where no free variable of A occurs in $\mathbf{x}$, if

$$\frac{A_1 \Longrightarrow B_1, \quad \ldots \quad , A_n \Longrightarrow B_n}{A_0 \Longrightarrow B_0} \in \Gamma \ ,$$

then

$$\Gamma \vdash \forall \mathbf{x}(A \wedge A_1 \to B_1) \wedge \cdots \wedge \forall \mathbf{x}(A \wedge A_n \to B_n) \Longrightarrow \forall \mathbf{x}(A \wedge A_0 \to B_0).$$

A theory Γ is *well-formed* if for all sequences of sentences $\forall \mathbf{x}(A_0 \to B_0), \forall \mathbf{x}(A_1 \to B_1), \ldots, \forall \mathbf{x}(A_n \to B_n)$ and formulas A where no free variable of A occurs in $\mathbf{x}$, if

$$\Gamma \cup \{A_1 \Longrightarrow B_1, \ldots, A_n \Longrightarrow B_n\} \vdash A_0 \Longrightarrow B_0,$$

then

$$\Gamma \vdash \forall \mathbf{x}(A \wedge A_1 \rightarrow B_1) \wedge \cdots \wedge \forall \mathbf{x}(A \wedge A_n \rightarrow B_n) \Longrightarrow \forall \mathbf{x}(A \wedge A_0 \rightarrow B_0).$$

A theory has a *well-formed axiomatization* if it is axiomatizable by a well-formed set.

Proposition 4.13 *A theory is well-formed if and only if it has a well-formed axiomatization.*

Proof: Well-formed theories are axiomatized by themselves, so they have well-formed axiomatizations. We prove the converse by induction on the complexity of derivations. Let Γ be a well-formed set, let $\forall \mathbf{x}(A_0 \rightarrow B_0), \forall \mathbf{x}(A_1 \rightarrow B_1), \ldots, \forall \mathbf{x}(A_n \rightarrow B_n)$ be a sequence of sentences, and let A be a formula of which no free variable occurs in $\mathbf{x}$, such that

$$\Gamma \cup \{A_1 \Longrightarrow B_1, \ldots, A_n \Longrightarrow B_n\} \vdash A_0 \Longrightarrow B_0.$$

If $\Gamma \vdash A_0 \Longrightarrow B_0$, or if $A_0 \Longrightarrow B_0$ equals $A_i \Longrightarrow B_i$ for some $i > 0$, then clearly $\Gamma \vdash X \Longrightarrow \forall \mathbf{x}(A \wedge A_0 \rightarrow B_0)$, where X is the required conjunction. It remains to check closure under the rules. The cases for the rules in Γ immediately follow from the definition of well-formed set. So we only have to check the rules of BQC. All proposition-logical ones are easy. Let $\mathbf{w}$ be the sequence of free variables of A, and let $\mathbf{z}$ be a sequence of new variables. Suppose that $B^{\mathbf{y}}_{\mathbf{t}} \Longrightarrow C^{\mathbf{y}}_{\mathbf{t}}$ follows from $B \Longrightarrow C$ by variable substitution. Then, with induction, $\Gamma \vdash X^{\mathbf{w}}_{\mathbf{z}} \Longrightarrow \forall \mathbf{x}'(A^{\mathbf{w}}_{\mathbf{z}} \wedge B \rightarrow C) \Longrightarrow \forall \mathbf{x}(A^{\mathbf{w}}_{\mathbf{z}} \wedge B^{\mathbf{y}}_{\mathbf{t}} \rightarrow C^{\mathbf{y}}_{\mathbf{t}})$, where the sequence $\mathbf{x}'$ includes all free variables of B and C. So $\Gamma \vdash X \Longrightarrow \forall \mathbf{x}(A \wedge B^{\mathbf{y}}_{\mathbf{t}} \rightarrow C^{\mathbf{y}}_{\mathbf{t}})$. Suppose that B equals $\exists y D$, and C doesn't have a free occurrence of y, such that $\exists y D \Longrightarrow C$ follows from $D \Longrightarrow C$. Then, with induction, $\Gamma \vdash X^{\mathbf{w}}_{\mathbf{z}} \Longrightarrow \forall \mathbf{x} y(A^{\mathbf{w}}_{\mathbf{z}} \wedge D \rightarrow C) \Longrightarrow \forall \mathbf{x}(A^{\mathbf{w}}_{\mathbf{z}} \wedge (\exists y D) \rightarrow C)$. So $\Gamma \vdash X \Longrightarrow \forall \mathbf{x}(A \wedge (\exists y D) \rightarrow C)$. Suppose $B \Longrightarrow C$ follows from $\exists y B \Longrightarrow C$. Then, with $\vdash \forall \mathbf{x}(B \rightarrow (\exists y B))$ and induction, $\Gamma \vdash X \Longrightarrow \forall \mathbf{x}(A \wedge (\exists y B) \rightarrow C) \Longrightarrow \forall \mathbf{x}(A \wedge B \rightarrow C)$. Finally, suppose C equals $\forall \mathbf{y}(D \rightarrow E)$, the variables in $\mathbf{y}$ are not free in B, and $B \Longrightarrow C$ follows from $B \wedge D \Longrightarrow E$. By induction, $\Gamma \vdash X^{\mathbf{w}}_{\mathbf{z}} \Longrightarrow \forall \mathbf{x}\mathbf{y}(A^{\mathbf{w}}_{\mathbf{z}} \wedge (B \wedge D) \rightarrow E)$. Since no variable in $\mathbf{y}$ is free in $A^{\mathbf{w}}_{\mathbf{z}} \wedge B$, BQC entails $A^{\mathbf{w}}_{\mathbf{z}} \wedge B \Longrightarrow \forall \mathbf{y}(D \rightarrow (A^{\mathbf{w}}_{\mathbf{z}} \wedge B \wedge D))$. So $A^{\mathbf{w}}_{\mathbf{z}} \wedge B \wedge \forall \mathbf{x}\mathbf{y}(A^{\mathbf{w}}_{\mathbf{z}} \wedge (B \wedge D) \rightarrow E) \Longrightarrow A^{\mathbf{w}}_{\mathbf{z}} \wedge B \wedge \forall \mathbf{y}(A^{\mathbf{w}}_{\mathbf{z}} \wedge B \wedge D \rightarrow E) \Longrightarrow \forall \mathbf{y}(D \rightarrow E)$. So BQC also proves $\forall \mathbf{x}\mathbf{y}(A^{\mathbf{w}}_{\mathbf{z}} \wedge (B \wedge D) \rightarrow E) \Longrightarrow \forall \mathbf{x}(A^{\mathbf{w}}_{\mathbf{z}} \wedge B \rightarrow \forall \mathbf{y}(D \rightarrow E))$. Apply transitivity of $\Longrightarrow$ and substitute $\mathbf{w}$ back for $\mathbf{z}$. □

Corollary 4.14 *BQC is well-formed. Sequent theory extensions of well-formed theories are well-formed.*

Well-formedness is a language-dependent property, but less so than functionality is. For if Γ is well-formed over $\mathcal{L}$, then Γ is a well-formed set over $\mathcal{L}[C]$, for all sets C of new constant symbols. We can also preserve well-formedness when we extend functional theories from $\mathcal{L}$ to $\mathcal{L}[C]$:

Proposition 4.15 *Let Γ be a functional well-formed set over $\mathcal{L}$, and C be a set of new constant symbols. Then $\Gamma\langle C\rangle$ is also a functional well-formed set.*

Proof: It suffices to establish well-formedness of $\Gamma\langle C\rangle$. Let $A \times R$ be a rule of $\Gamma\langle C\rangle \setminus \Gamma$, where

$$R = \frac{A_1 \Longrightarrow B_1 \quad \dots \quad A_n \Longrightarrow B_n}{A_0 \Longrightarrow B_0}$$

is a rule of Γ whose free variables all occur in $\mathbf{x}$, and A is a sentence of $\mathcal{L}[C]$. Let $B \in \mathcal{L}[C]$ be a formula whose free variables do not occur in $\mathbf{x}$. To prove

$$\Gamma\langle C\rangle \vdash \forall\mathbf{x}(B \wedge A \wedge A_1 \to B_1) \wedge \cdots \wedge \\ \forall\mathbf{x}(B \wedge A \wedge A_n \to B_n) \Longrightarrow \forall\mathbf{x}(B \wedge A \wedge A_n \to B_n).$$

Let $\mathbf{c}$ include all new constant symbols in $B \wedge A$, and let $\mathbf{z}$ be a new variable sequence of equal length. As Γ is well-formed and $R \in \Gamma$,

$$\Gamma \vdash \forall\mathbf{x}((B \wedge A)^{\mathbf{c}}_{\mathbf{z}} \wedge A_1 \to B_1) \wedge \cdots \wedge \\ \forall\mathbf{x}((B \wedge A)^{\mathbf{c}}_{\mathbf{z}} \wedge A_n \to B_n) \Longrightarrow \forall\mathbf{x}((B \wedge A)^{\mathbf{c}}_{\mathbf{z}} \wedge A_n \to B_n).$$

Substitute $\mathbf{c}$ in for $\mathbf{z}$. □

5 Kripke models with functions, and strong completeness Kripke models $\mathbf{K} = \langle \mathbf{D}^{\mathbf{K}}, I^{\mathbf{K}}\rangle$ for languages with function models are constructed as in Section 3, but with the following extensions.

To each function symbol f, I assigns a natural transformation $I(f) = F_f : \mathbf{D}^n \to \mathbf{D}$ in the presheaf category over $(W, \preceq)$. So for all α we have a function $F_f\alpha : (D\alpha)^n \to D\alpha$, and for each pair $\alpha \preceq \beta$ a map $(F_f)^\alpha_\beta : D\alpha \to D\beta$ such that the diagram

$$\begin{array}{ccc} (D\alpha)^n & \xrightarrow{F_f\alpha} & D\alpha \\ \big\downarrow{\scriptstyle (D^\alpha_\beta)^n} & & \big\downarrow{\scriptstyle D^\alpha_\beta} \\ (D\beta)^n & \xrightarrow{F_f\beta} & D\beta \end{array}$$

commutes, for all pairs $\alpha \preceq \beta$. The interpretation of functions f is extended to terms t by setting $I(t)$ equal to the usual composition T_t of the interpretations of the parts that make up the term. Constants c are 0-ary functions, so $I(c)$ essentially consists of a collection of elements $\{d_\alpha \in D\alpha\}_{\alpha\in W}$ such that $D^\alpha_\beta d_\alpha = d_\beta$ whenever $\alpha \preceq \beta$. This agrees with the interpretation for constants in Section 3.

As in Section 3 we form an extended language $\mathcal{L}[C\alpha]$ of the original language $\mathcal{L}$ for each node α by adding a unique set of constant symbols $C\alpha \cong D\alpha$. For each α the map I_α on the set of (constant and) function symbols of $\mathcal{L}[C\alpha]$ assigns to each function symbol f of $\mathcal{L}$, the function $I_\alpha(f) = F_f\alpha$, and to each new constant symbol $c_\alpha \in C\alpha$ its corresponding element $I_\alpha(c_\alpha) = d_\alpha \in D\alpha$. A term is *closed* if it does not contain variables. The set of closed terms of $\mathcal{L}[C\alpha]$ is called $T\alpha$. We extend I_α to all closed terms $T\alpha$ in the obvious way. For all terms $t_\alpha \in T\alpha$ we have $I_\beta(t_\beta) = D^\alpha_\beta I_\alpha(t_\alpha)$.

Let $\mathbf{t} = (t_1, \dots, t_n)$ be a sequence of n closed terms of $\mathcal{L}[C\alpha]$, and let P be an n-ary predicate. Then we write $\alpha \Vdash P(\mathbf{t})$ if $I_\alpha(\mathbf{t}) = (I_\alpha(t_1), \dots, I_\alpha(t_n)) \in R_P\alpha \subseteq$

$(D\alpha)^n$. The relation $\Vdash$ is uniquely extended to all sentences by the inductive definition as presented in Section 3, except that we choose these modifications:

1. $\alpha \Vdash \exists x A$ if and only if there exist $t \in T\alpha$ such that $\alpha \Vdash A^x_t$; and
2. $\alpha \Vdash \forall \mathbf{x}(A_\alpha \to B_\alpha)$ if and only if for all $\beta \succ \alpha$ and $\mathbf{t} \in (T\beta)^n$, $\beta \Vdash (A_\beta)^{\mathbf{x}}_{\mathbf{t}}$ implies $\beta \Vdash (B_\beta)^{\mathbf{x}}_{\mathbf{t}}$.

One easily proves persistence: if $\beta \succeq \alpha \Vdash A_\alpha$, then $\beta \Vdash A_\beta$. We can extend the relation $\Vdash$ in a natural way to all formulas by $\alpha \Vdash A_\alpha$ if and only if for all $\beta \succeq \alpha$ and $\mathbf{t} \in (T\beta)^n$ we have $\beta \Vdash (A_\beta)^{\mathbf{x}}_{\mathbf{t}}$, where $\mathbf{x} = (x_1, \ldots, x_n)$ includes all free variables of A_α. For formulas $A_\alpha \to B_\alpha$ where $\mathbf{x}$ includes all free variables, this implies $\alpha \Vdash A_\alpha \to B_\alpha$ if and only if for all $\gamma \succ \beta \succeq \alpha$ and $\mathbf{t} \in (T\beta)^n$, $\gamma \Vdash ((A_\beta)^{\mathbf{x}}_{\mathbf{t}})_\gamma$ implies $\gamma \Vdash ((B_\beta)^{\mathbf{x}}_{\mathbf{t}})_\gamma$.

We extend $\Vdash$ to all sequents by

$\alpha \Vdash A_\alpha \Longrightarrow B_\alpha$ iff for all $\beta \succeq \alpha$ and $\mathbf{t} \in (T\beta)^n$, $\beta \Vdash (A_\beta)^{\mathbf{x}}_{\mathbf{t}}$ implies $\beta \Vdash (B_\beta)^{\mathbf{x}}_{\mathbf{t}}$.

As usual, $\alpha \Vdash A$ if and only if $\alpha \Vdash\Longrightarrow A$. Extend $\Vdash$ to rules in the usual way.

Proposition 5.1 *Given a formula A with all its free variables among $\mathbf{x}$, and a node α, let*

$$R_A = \{\mathbf{d} \in (D\alpha)^n \mid \alpha \Vdash A^{\mathbf{x}}_{I_\alpha^{-1}(\mathbf{d})}\}.$$

Then $\alpha \Vdash A^{\mathbf{x}}_{\mathbf{t}}$ if and only if $I_\alpha(\mathbf{t}) \in R_A$, for all $\mathbf{t} \in (T\alpha)^n$.

Proof: The case for atomic A follows from the definitions. The general case easily follows by induction on the complexity of A. □

Proposition 5.1 immediately implies that for all $t \in T\alpha$ there is a unique $c_\alpha \in C\alpha$ such that $\alpha \Vdash t = c_\alpha$. So the inductive conditions on both quantification and on sequents is equivalent to the ones in Section 3. So the new forcing definitions agree with the versions in Section 3. Because of this almost-identity between constant symbols and constants, it is common to write $A^{\mathbf{x}}_{\mathbf{d}}$ as short for

$$A^{\mathbf{x}}_{I_\alpha^{-1}\mathbf{d}}$$

if confusion is unlikely. An immediate consequence of Proposition 5.1 is

Proposition 5.2 *For all formulas A,*

$$\alpha \Vdash x = y \wedge A \Longrightarrow A^x_y,$$

where y does not become bound after substitution.

The satisfaction relation $\models$ is defined as usual.

Proposition 5.3 (Soundness) *Let Γ be a set of sequents and rules, and γ be a sequent. Then $\Gamma \vdash \gamma$ implies $\Gamma \models \gamma$. If R is a rule such that $\Gamma \vdash R$, then $\Gamma \models R$.*

Proof: The case for rules easily follows from the case for sequents. It suffices to show that $\Vdash$ satisfies the axiom sequents of BQC and is closed under its rules. For the proposition-logical fragment, see [1]. Suppose $\alpha \Vdash A_\alpha \Longrightarrow B_\alpha$ with all free variables among $\mathbf{x}$, and let $\mathbf{t}$ be a sequence of terms with all free variables among $\mathbf{y}$, none of whom is bound by a quantifier after substitution in $(A_\alpha)^{\mathbf{x}}_{\mathbf{t}}$ or in $(B_\alpha)^{\mathbf{x}}_{\mathbf{t}}$. Now let $\beta \succeq \alpha$ and $\mathbf{s} \in (T\beta)^n$ be such that $\beta \Vdash ((A_\beta)^{\mathbf{x}}_{\mathbf{t}})^{\mathbf{y}}_{\mathbf{s}}$. Now $\beta \Vdash (A_\beta)^{\mathbf{x}}_{\mathbf{u}}$ implies $\beta \Vdash (B_\beta)^{\mathbf{x}}_{\mathbf{u}}$, for all terms $\mathbf{u} \in (T\beta)^m$ so, in particular, $\beta \Vdash ((B_\beta)^{\mathbf{x}}_{\mathbf{t}})^{\mathbf{y}}_{\mathbf{s}}$. Suppose that $\alpha \Vdash B_\alpha \Longrightarrow A_\alpha$, where x and $\mathbf{y}$ include all free variables, x not in $\mathbf{y}$, and x is not free in A_α. Let $\beta \succeq \alpha$ and $\mathbf{t} \in (T\beta)^n$ be such that $\beta \Vdash \exists x (B_\beta)^{\mathbf{y}}_{\mathbf{t}}$. Then $\beta \Vdash (B_\beta)^{x,\mathbf{y}}_{t,\mathbf{t}}$ for some $t \in T\beta$, so $\beta \Vdash (A_\beta)^{\mathbf{y}}_{\mathbf{t}}$. Suppose that $\alpha \Vdash \exists x B_\alpha \Longrightarrow A_\alpha$, and let $\beta \succeq \alpha$ and $t \in T\beta$ and $\mathbf{t} \in (T\beta)^n$ be such that $\beta \Vdash (B_\beta)^{x,\mathbf{y}}_{t,\mathbf{t}}$. Then $\beta \Vdash \exists x (B_\beta)^{\mathbf{y}}_{\mathbf{t}}$, so $\beta \Vdash (A_\beta)^{\mathbf{y}}_{\mathbf{t}}$. Suppose that A_α, B_α, and C_α are formulas with all free variables among $\mathbf{xy}$, and no variable of $\mathbf{y}$ free in A_α, such that $\alpha \Vdash A_\alpha \wedge B_\alpha \Longrightarrow C_\alpha$, and let $\gamma \succ \beta \succeq \alpha$, $\mathbf{t} \in (T\beta)^m$, and $\mathbf{u} \in (T\gamma)^n$ be such that $\beta \Vdash (A_\beta)^{\mathbf{x}}_{\mathbf{t}}$ and $\gamma \Vdash ((B^{\mathbf{x}}_{\mathbf{t}})_\gamma)^{\mathbf{y}}_{\mathbf{u}}$. Then $\gamma \Vdash (A^{\mathbf{x}}_{\mathbf{t}})_\gamma \wedge ((B^{\mathbf{x}}_{\mathbf{t}})_\gamma)^{\mathbf{y}}_{\mathbf{u}}$, and thus $\gamma \Vdash ((C^{\mathbf{x}}_{\mathbf{t}})_\gamma)^{\mathbf{y}}_{\mathbf{u}}$. So $\beta \Vdash \forall \mathbf{y}((B_\beta)^{\mathbf{x}}_{\mathbf{t}} \to (C_\beta)^{\mathbf{x}}_{\mathbf{t}})$. □

For each model $\mathbf{K}$ there is a theory $\mathrm{Th}(\mathbf{K})$ of all rules and sequents that hold in $\mathbf{K}$. Each node α of a Kripke model $\mathbf{K}$ yields a model $\mathbf{K}_\alpha$ in the usual way, by restricting the collection of worlds to exactly the nodes β such that $\beta \succeq \alpha$. So $\alpha \Vdash \gamma$ if and only if $\gamma \in \mathrm{Th}(\mathbf{K}_\alpha)$, for all sequents γ, similarly for rules. Then $\mathrm{Th}(\mathbf{K}) = \cap_\alpha \mathrm{Th}(\mathbf{K}_\alpha)$, where α ranges over all nodes of $\mathbf{K}$. Kripke models with minimal nodes are called *rooted*.

Let D be a set of constant symbols. A theory Γ is *D-saturated* if it satisfies the usual conditions as in Section 3. A set of sequents and rules is *D-saturated* if it axiomatizes a D-saturated theory.

Proposition 5.4 *For all Kripke models* $\mathbf{K}$ *the theory Th*$(\mathbf{K})$ *is a functional well-formed theory. If* $\mathbf{K}$ *is rooted with root* α_0*, then Th*$(\mathbf{K})$ *is* $C\alpha_0$*-saturated.*

Proof: Suppose that

$$R = \frac{A_1 \Longrightarrow B_1 \quad \dots \quad A_n \Longrightarrow B_n}{A_0 \Longrightarrow B_0} \in \mathrm{Th}(\mathbf{K}),$$

and let α be a node, and A be a sentence over $\mathcal{L}[C\alpha]$, such that $\alpha \Vdash A \wedge A_i \Longrightarrow B_i$ for all $i > 0$, and $\alpha \Vdash A \wedge A_0$. Then $\alpha \Vdash A_i \Longrightarrow B_i$, for all $i > 0$, so $\alpha \Vdash B_0$. So $\mathrm{Th}(\mathbf{K})$ is functional.

There exists a sequence of variables $\mathbf{x}$ such that all free variables of R occur in $\mathbf{x}$. Let B be a formula with no free variables in $\mathbf{x}$, and $\alpha \Vdash \forall \mathbf{x}(B \wedge A_i \to B_i)$ for all $i > 0$. Let $\beta \succ \alpha$ be such that $\beta \Vdash (B \wedge A_0)^{\mathbf{y}}_{\mathbf{c}}$, where $\mathbf{y}$ includes all free variables of $B \wedge A_0 \wedge B_0$, and the elements of $\mathbf{c}$ are constants from $C\beta$. But $\beta \Vdash B^{\mathbf{y}}_{\mathbf{c}} \wedge A_i \Longrightarrow B_i$ for all $i > 0$, so $\beta \Vdash (B_0)^{\mathbf{y}}_{\mathbf{c}}$. So $\alpha \Vdash \forall \mathbf{x}(B \wedge A_0 \to B_0)$. So $\mathrm{Th}(\mathbf{K})$ is well-formed.

$C\alpha_0$-saturation for rooted models is just as easy. □

We will show below that all functional well-formed theories are of the form $\mathrm{Th}(\mathbf{K})$.

Lemma 5.5 *Let* Γ *be a functional well-formed theory over a language* $\mathcal{L}$*, and* $\gamma \in \mathcal{L}$ *be a sequent such that* $\Gamma \nVdash \gamma$*. Let* C *be a set of new constant symbols of cardinality equal to the cardinality of* $\mathcal{L}$*. Then there is a* C*-saturated functional well-formed theory* $\Delta \supseteq \Gamma$ *over* $\mathcal{L}[C]$ *such that* $\Delta \nVdash \gamma$*.*

Proof: Partition C into a countably infinite sequence of subsets $C_1, C_2, C_3, \dots$, all of cardinality equal to the cardinality of $\mathcal{L}$, and let $T_i = C_1 \cup \dots \cup C_i$, for all i. We construct a sequence of theories $\Gamma \subseteq \Delta_1 \subseteq \Gamma_1 \subseteq \Delta_2 \subseteq \Gamma_2 \subseteq \cdots$ such that for all i

1. Δ_i and Γ_i are functional well-formed theories over $\mathcal{L}[T_i]$;

2. $\Delta_i \nvdash \gamma$, $\Gamma_i \nvdash \gamma$;
3. for all sentences $\exists x A$ over $\mathcal{L}[T_{i-1}]$, if $\Delta_i \vdash \exists x A$, then $\Delta_i \vdash A^x_t$, for some t; and
4. for all sentences A and B over $\mathcal{L}[T_i]$, if $\Gamma_i \vdash A \vee B$, then $\Gamma_i \vdash A$ or $\Gamma_i \vdash B$.

Given Δ_i, we construct the extension Γ_i in the usual way using Zorn's Lemma (see [1]). So Γ_i is a maximal sequent theory extension of Δ_i satisfying $\Gamma_i \nvdash \gamma$. By Corollaries 4.10 and 4.14, Γ_i is functional and well-formed.

Let Γ_0 equal Γ. Given Γ_{i-1}, we construct Δ_i in two stages. First consider $\Gamma_{i-1}\langle C_i\rangle$. This is a functional well-formed theory such that $\Gamma_{i-1}\langle C_i\rangle \nvdash \gamma$. Second, there is a bijective function which assigns to each existential sentence of $\mathcal{L}[T_{i-1}]$ for which $\Gamma_{i-1}\langle C_i\rangle \vdash \exists x A$, a constant symbol $c(A)$ from the set of new constant symbols C_i. Set Δ_i equal to the theory over $\mathcal{L}[T_i]$ axiomatized by $\Gamma_{i-1}\langle C_i\rangle$ plus all such sentences $A^x_{c(A)}$. So Δ_i is functional and well-formed. Suppose $\Delta_i \vdash \exists y B$, for some $B \in \mathcal{L}[T_{i-1}]$. Then there is a sentence $\exists x_1 A_1 \wedge \cdots \wedge \exists x_n A_n = \exists \mathbf{x} A$ over $\mathcal{L}[T_{i-1}]$, and a sequence $\mathbf{c} \in (C_i)^n$ such that $\Gamma_{i-1}\langle C_i\rangle \vdash \exists \mathbf{x} A$ and $\Gamma_{i-1}\langle C_i\rangle \cup \{A^{\mathbf{x}}_{\mathbf{c}}\} \vdash \exists y B$. So, by functional completeness, $\Gamma_{i-1}\langle C_i\rangle \vdash A^{\mathbf{x}}_{\mathbf{c}} \Longrightarrow \exists y B$. Let $\mathbf{z}$ be a sequence of new variables. Then, by functional generalization, $\Gamma_{i-1}\langle C_i\rangle \vdash A^{\mathbf{x}}_{\mathbf{z}} \Longrightarrow \exists y B$, hence also $\Gamma_{i-1}\langle C_i\rangle \vdash \exists \mathbf{z} A^{\mathbf{x}}_{\mathbf{z}} \Longrightarrow \exists y B$. But then, with renaming of bound variables, $\Gamma_{i-1}\langle C_i\rangle \vdash \exists y B$, and thus $\Delta_i \vdash B^y_{c(B)}$. Let $B \Longrightarrow C$ be a sequent over $\mathcal{L}[T_{i-1}]$ such that $\Delta_i \vdash B \Longrightarrow C$. Then there is a sentence $\exists \mathbf{x} A$ over $\mathcal{L}[T_{i-1}]$, and a sequence $\mathbf{c} \in (C_i)^n$ such that $\Gamma_{i-1}\langle C_i\rangle \vdash \exists \mathbf{x} A$ and $\Gamma_{i-1}\langle C_i\rangle \cup \{A^{\mathbf{x}}_{\mathbf{c}}\} \vdash B \Longrightarrow C$. So $\Gamma_{i-1}\langle C_i\rangle \vdash A^{\mathbf{x}}_{\mathbf{c}} \wedge B \Longrightarrow C$. Let $\mathbf{z}$ be a sequence of new variables. Then $\Gamma_{i-1}\langle C_i\rangle \vdash A^{\mathbf{x}}_{\mathbf{z}} \wedge B \Longrightarrow C$, hence also $\Gamma_{i-1}\langle C_i\rangle \vdash \exists \mathbf{z} A^{\mathbf{x}}_{\mathbf{z}} \wedge B \Longrightarrow C$, and thus $\Gamma_{i-1}\langle C_i\rangle \vdash B \Longrightarrow C$. So, in particular, $\Delta_i \nvdash \gamma$. This completes the construction of the sequence of theories. Set $\Delta = \cup_i \Delta_i$. Then Δ is a C-saturated functional well-formed extension of Γ over $\mathcal{L}[C]$ such that $\Delta \nvdash \gamma$. □

Given a language $\mathcal{L}$, we construct a universal Kripke model $\mathbf{U} = \mathbf{U}_{\mathcal{L}}$ as follows. Let C be a set of constant symbols of cardinality equal to the cardinality of $\mathcal{L}$. Partition C into a countably infinite sequence of subsets $C_1, C_2, C_3, \ldots$, all of cardinality equal to the cardinality of $\mathcal{L}$, and let $T_i = C_1 \cup \cdots \cup C_i$, for all i. As set of worlds $W = W^{\mathbf{U}}$ we choose the collection of all T_i-saturated functional well-formed theories over $\mathcal{L}[T_i]$, for all i. We write $\Gamma \prec \Delta$ when $\Gamma \vdash \forall \mathbf{x}(A \to B)$ implies $\Delta \vdash A \Longrightarrow B$, for all sentences $\forall \mathbf{x}(A \to B)$. So $\prec$ is obviously transitive. Given a theory Γ over $\mathcal{L}$, define $\Gamma^{(1)}$ to be the theory over $\mathcal{L}$ axiomatized by Γ plus $\{A \Longrightarrow B \mid \forall \mathbf{x}(A \to B)$ is a sentence and $\Gamma \vdash \forall \mathbf{x}(A \to B)\}$. Then $\Gamma^{(1)}$ is a sequent theory extension of Γ, hence functional and well-formed if Γ is. Lemma 5.5 implies that $\Gamma^{(1)} = \mathcal{L} \cap \bigcap\{\Delta \in W \mid \Gamma \prec \Delta\}$, for all nodes Γ. For each node Γ, let $T\Gamma$ be its set of constant symbols. The relation $s \sim t$, defined by $\Gamma \vdash s = t$, is an equivalence relation on $T\Gamma$. The equivalence class of a constant symbol s is denoted by $[s] = [s]_\Gamma$. Set $D\Gamma$ equal to the set of equivalence classes $T\Gamma/\sim$. If $\Gamma \prec \Delta$, then $T\Gamma \subseteq T\Delta$, and $\Gamma \subseteq \Delta$ implies that $[s]_\Gamma \subseteq [s]_\Delta$, for all $s \in T\Gamma$. So the inclusion map $T\Gamma \to T\Delta$ induces a function $D^\Gamma_\Delta : D\Gamma \to D\Delta$ such that D^Γ_Γ is the identity, and $D^\Delta_\Psi D^\Gamma_\Delta = D^\Gamma_\Psi$, whenever $\Gamma \prec \Delta \prec \Psi$. This completes the construction of the Kripke structure. The structure is inhabited, since BQC satisfies $\exists x\, x = x$. It remains to define the interpretation map $I = I^{\mathbf{U}}$. Let P be an n-ary predicate, Γ be a node, and let $X = \{\mathbf{s} \in (T\Gamma)^n \mid \Gamma \vdash P\mathbf{s}\}$. Then X is a subset of $(T\Gamma)^n$ such that if $\mathbf{s} \in X$

and $[\mathbf{s}] = ([s_1], \ldots, [s_n]) = [\mathbf{t}]$, then $\mathbf{t} \in X$. So $I(P)_\Gamma = X/\sim$ is a well-defined subset of $(D\Gamma)^n$. If $\Gamma \prec \Delta$, then $D^\Gamma_\Delta(I(P)_\Gamma) \subseteq I(P)_\Delta$, so I(P) can be (uniquely) extended to a substructure of $\mathbf{D}^n$. Obviously, if P is the equality symbol, then $I(P)$ is the diagonal substructure of $\mathbf{D}^2$. Let f be a function symbol. Since BQC satisfies $\mathbf{x} = \mathbf{y} \Longrightarrow f\mathbf{x} = f\mathbf{y}$, the assignment $I_\Gamma(f)([\mathbf{s}]) = [f\mathbf{s}]$ is a well-defined map from $(D\Gamma)^n$ to $D\Gamma$. As $\Gamma \prec \Delta$ implies $\Gamma \subseteq \Delta$, the diagram

$$\begin{array}{ccc} (D\Gamma)^n & \xrightarrow{\;I_\Gamma(f)\;} & D\Gamma \\ {\scriptstyle (D^\Gamma_\Delta)^n}\downarrow & & \downarrow{\scriptstyle D^\Gamma_\Delta} \\ (D\Delta)^n & \xrightarrow{\;I_\Delta(f)\;} & D\Delta \end{array}$$

commutes. So $I(f)$ is a natural transformation from $\mathbf{D}^n$ to $\mathbf{D}$.

Lemma 5.6 *Let Γ be a well-formed theory. Then $\Gamma^{(1)} \vdash A \Longrightarrow B$ if and only if $\Gamma \vdash \forall\mathbf{x}(A \to B)$, for all sentences $\forall\mathbf{x}(A \to B)$.*

Proof: From right to left immediately follows from the definition. The converse follows from the well-formedness of Γ. □

Lemma 5.7 *For all $\Gamma \in W^{\mathbf{U}}$ and all sentences A over $\mathcal{L}[T\Gamma]$ we have $\Gamma \vdash A$ if and only if $\Gamma \Vdash A$. For all sequents $A \Longrightarrow B$ over $\mathcal{L}[T\Gamma]$, $\Gamma \vdash A \Longrightarrow B$ implies $\Gamma \Vdash A \Longrightarrow B$.*

Proof: We complete the proof of the sentence case by induction on the complexity of A. The cases where A equals $\top$ or $\bot$ are trivial. The case for A an atomic sentence immediately follows from the definitions. Suppose the sentence A equals $B \vee C$. Then, since Γ is saturated, $\Gamma \vdash B \vee C$ if and only if $\Gamma \vdash B$ or $\Gamma \vdash C$; apply induction. The case for A equal to $B \wedge C$ is trivial. Suppose the sentence A equals $\exists x B$. Then, since Γ is saturated, $\Gamma \vdash \exists x B$ if and only if $\Gamma \vdash Bt$ for some term t; apply induction. Suppose the sentence A equals $\forall\mathbf{x}(B \to C)$. Let $\Delta \succ \Gamma \vdash \forall\mathbf{x}(B \to C)$ and $\mathbf{s} \in (T\Delta)^n$ be such that $\Delta \Vdash B^{\mathbf{x}}_{\mathbf{s}}$. Then $\Delta \vdash B \Longrightarrow C$ and, by induction, $\Delta \vdash B^{\mathbf{x}}_{\mathbf{s}}$. So $\Delta \vdash C^{\mathbf{x}}_{\mathbf{s}}$ so, by induction, $\Delta \Vdash C^{\mathbf{x}}_{\mathbf{s}}$. Thus $\Gamma \Vdash \forall\mathbf{x}(B \to C)$. Conversely, suppose $\Gamma \Vdash \forall\mathbf{x}(B \to C)$. Let $\Delta \succ \Gamma$ be such that $\Delta \nvdash B \Longrightarrow C$. Then $\Delta \supseteq \Gamma^{(1)}$, and, by functional generalization, for new constant symbols $\mathbf{s}$ we have $\Delta\langle\mathbf{s}\rangle \nvdash B^{\mathbf{x}}_{\mathbf{s}} \Longrightarrow C^{\mathbf{x}}_{\mathbf{s}}$. Apply functional completeness: $\Delta \cup \{B^{\mathbf{x}}_{\mathbf{s}}\} \nvdash C^{\mathbf{x}}_{\mathbf{s}}$. By Lemma 5.5 there is a node $\Psi \supseteq \Delta\langle\mathbf{s}\rangle \cup \{B^{\mathbf{x}}_{\mathbf{s}}\}$ such that $\Psi \nvdash C^{\mathbf{x}}_{\mathbf{s}}$. But $\Psi \supseteq \Gamma^{(1)}$, so $\Psi \succ \Gamma$ is such that, by induction, $\Psi \Vdash B^{\mathbf{x}}_{\mathbf{s}}$ and $\Psi \nVdash C^{\mathbf{x}}_{\mathbf{s}}$; contradiction. So $\Delta \vdash B \Longrightarrow C$. Therefore $\Gamma^{(1)} = \bigcap\{\Delta \in W \mid \Gamma \prec \Delta\} \vdash B \Longrightarrow C$. And thus $\Gamma \vdash \forall\mathbf{x}(B \to C)$.

As to the sequent case, suppose $\Delta \succeq \Gamma \vdash A \Longrightarrow B$ and $\mathbf{s} \in (T\Delta)^n$ are such that $\Delta \Vdash A^{\mathbf{x}}_{\mathbf{s}}$. Then, by the sentence case above, $\Gamma \subseteq \Delta \vdash A^{\mathbf{x}}_{\mathbf{s}}$, and so $\Delta \vdash B^{\mathbf{x}}_{\mathbf{s}}$. Thus $\Gamma \Vdash A \Longrightarrow B$. □

As shown in [1], the sequent case of Lemma 5.7 cannot be extended to an equivalence.

For each prime theory Δ over some $\mathcal{L}[T_i]$, the set of all nodes $\Psi \succeq \Delta$ generates a submodel $\mathbf{U}_\Delta$ of $\mathbf{U}$ with root Δ. So for all sequents γ, $\mathbf{U}_\Delta \models \gamma$ if and only if $\Delta \Vdash \gamma$, similarly for rules.

Theorem 5.8 (Completeness for functional well-formed theories) *Let Γ be a functional well-formed theory, and γ be a sequent. Then $\Gamma \models \gamma$ implies $\Gamma \vdash \gamma$. If R is a rule such that $\Gamma \models R$, then $\Gamma \vdash R$.*

Proof: Let γ be the sequent $A \Longrightarrow B$ such that $\Gamma \nvdash A \Longrightarrow B$. Let $\mathbf{s}$ be a sequent of new constant symbols from S. Then, by functional generalization and functional completeness, $\Gamma\langle\mathbf{s}\rangle \cup \{A^{\mathbf{x}}_{\mathbf{s}}\} \nvdash B^{\mathbf{x}}_{\mathbf{s}}$. By Lemma 5.5 there exists a node $\Delta \supseteq \Gamma\langle\mathbf{s}\rangle \cup \{A^{\mathbf{x}}_{\mathbf{s}}\}$ such that $\Delta \nvdash B^{\mathbf{x}}_{\mathbf{s}}$. Then $\mathbf{U}_\Delta \models \Gamma$, but $\mathbf{U}_\Delta \not\models \gamma$.

Suppose $\Gamma \nvdash R$, where

$$R = \frac{\gamma_1 \quad \dots \quad \gamma_n}{\gamma}.$$

Then $\Gamma \cup \{\gamma_1, \dots, \gamma_n\} \nvdash \gamma$. So, by the sequent case above, $\Gamma \cup \{\gamma_1, \dots, \gamma_n\} \not\models \gamma$. So $\Gamma \not\models R$. □

Corollary 5.9 *All functional well-formed theories are of the form Th($\mathbf{K}$).*

Proof: If $\{\mathbf{K}_i\}_i$ is a collection of disjoint models, then $\mathrm{Th}(\cup_i \mathbf{K}_i) = \cap_i \mathrm{Th}(\mathbf{K}_i)$. □

A set of sequents and rules Γ is *complete* with respect to a class of Kripke models $\mathcal{K}$, if for all sequents $A \Longrightarrow B$ we have $\Gamma \vdash A \Longrightarrow B$, if and only if $\mathbf{K} \models A \Longrightarrow B$ for all $\mathbf{K} \in \mathcal{K}$. So a theory is complete for a class of models if and only if it is functional and well-formed.

The set Γ is *strongly complete* with respect to a class $\mathcal{K}$ of models, if Γ is complete with respect to $\mathcal{K}$, and if, moreover, for all functional well-formed sets $\Delta \supseteq \Gamma$ there is a subclass of models of $\mathcal{K}$ such that Δ is complete with respect to the subclass. So BQC is strongly complete with respect to the class of all Kripke models.

A Kripke model is a *tree model* if the reflexive closure $(W, \preceq)$ of the underlying set of nodes is a partially ordered set such that the predecessors of each node form a finite linear set. A Kripke model is *irreflexive* if $\prec$ is irreflexive.

Theorem 5.10 (Strong completeness for irreflexive trees) *BQC is strongly complete with respect to the class of irreflexive Kripke tree models.*

Proof: Form a new Kripke model $\mathbf{V}$ from $\mathbf{U}$ by redefining $\Gamma \prec \Delta$ if and only if $\Gamma^{(1)} \subseteq \Delta$ and, additionally, Γ and Δ are over languages $\mathcal{L}[T_i]$ and $\mathcal{L}[T_j]$ with $i < j$. Then all relevant lemmas above hold for $\mathbf{V}$ as for $\mathbf{U}$, with no need for a change in proofs except for replacing $\mathbf{U}$ by $\mathbf{V}$. But the models $\mathbf{V}_\Delta$ are irreflexive Kripke tree models. □

What is the connection between the two completeness theorems in this paper, other than Theorem 5.10 being a generalization of Theorem 3.5? Theorem 3.5 and its proof are almost identical to the case where we permit sequent theories. So the real difference is with the limited permission of rules in Theorem 5.10. The following model transformation technique connects the two versions.

Given a Kripke model $\mathbf{K}$, let $\mathbf{V}(\mathbf{K})$ be the model formed from $\mathbf{K}$ by adding, for each irreflexive node α, a new irreflexive node $\alpha' \prec \alpha$, with $D\alpha' = D\alpha$, and such that $\alpha' \Vdash A$ exactly when $\alpha \Vdash A$, for all atomic sentences A of $\mathcal{L}[C\alpha]$. Note that $\alpha' \prec \beta$ exactly when $\alpha \preceq \beta$. The following construction is more general: let S be a function which assigns to each irreflexive node α a subset $S\alpha$ of $\{A \mid$

A is an atom of $\mathcal{L}[C\alpha]$ and $\alpha \Vdash A\}$. Let $T\alpha'$ be the collection of terms constructed from the constant symbols $C\alpha$. The equivalence relation $s \sim t$ on $T\alpha'$ is defined by $S\alpha \vdash s = t$. Set the domain $D\alpha' = T\alpha'/\sim$. Then $\mathbf{V}_S(\mathbf{K})$ is the extension of $\mathbf{K}$ with the same new nodes as $\mathbf{V}(\mathbf{K})$, but $\alpha' \Vdash A$ if and only if $S\alpha \vdash A$, for all atoms A.

Let R be the rule

$$R = \frac{A_1 \Longrightarrow B_1 \quad \ldots \quad A_n \Longrightarrow B_n}{A_0 \Longrightarrow B_0}\ ,$$

and $\mathbf{x}$ the sequence of all free variables of R. Then we define the sequent $R^{(-1)}$ by

$$\forall\mathbf{x}(A_1 \to B_1) \wedge \cdots \wedge \forall\mathbf{x}(A_n \to B_n) \Longrightarrow \forall\mathbf{x}(A_0 \to B_0).$$

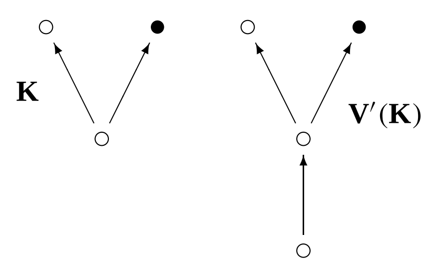
Sequents themselves are rules, but with empty numerator. So if γ is the sequent $A \Longrightarrow B$ with $\mathbf{x}$ its sequence of free variables, then $\gamma^{(-1)}$ equals $\forall\mathbf{x}(A \to B)$. If Γ is a set of rules (and sequents), then $\Gamma^{(-1)} = \{R^{(-1)} \mid R \in \Gamma\}$. If Γ is a well-formed theory, then $\Gamma^{(-1)} \subseteq \Gamma$.

Lemma 5.11 *For all Kripke models* $\mathbf{K}$, *all assignments* S, *and all rules* R, $\mathbf{K} \models R$ *if and only if* $\mathbf{V}_S(\mathbf{K}) \models R^{(-1)}$.

Proof: Let W be the set of nodes of $\mathbf{K}$, and W' the set of nodes of $\mathbf{V}_S(\mathbf{K})$ that are not in W. Suppose $\mathbf{K} \models R$. Then $\mathbf{K} \models R^{(-1)}$. So $\alpha \Vdash R^{(-1)}$, for all $\alpha \in W$. Let $\alpha' \in W'$. Then α' is irreflexive, and $\alpha' \prec \beta$ implies $\beta \in W$. So $\alpha' \Vdash \forall\mathbf{x}(A \to B)$ if and only if $\beta \Vdash A \Longrightarrow B$ for all $\beta \succ \alpha'$, for all sentences $\forall\mathbf{x}(A \to B)$. So $\alpha' \Vdash R^{(-1)}$, if and only if $\beta \Vdash R$ for all $\beta \succ \alpha'$. But all $\beta \succ \alpha'$ are in W. So $\alpha' \Vdash R^{(-1)}$. Conversely, suppose $\mathbf{V}_S(\mathbf{K}) \models R^{(-1)}$. If $\beta \in W$ is such that $\beta \succ \alpha' \in W'$ for some α', then $\beta \Vdash R$. Otherwise, β is reflexive and $\beta \prec \beta \Vdash R$. □

Let $\mathbf{K}$ be a rooted model with irreflexive root α, and let $\mathbf{V}'(\mathbf{K})$ be constructed from $\mathbf{K}$ just as $\mathbf{V}(\mathbf{K})$, but by only adding a new irreflexive node α' below the root. Similarly, construct $\mathbf{V}'_S(\mathbf{K})$ from $\mathbf{K}$. The following counterexample shows that the preceding lemma fails when we replace $\mathbf{V}_S(\mathbf{K})$ by $\mathbf{V}'(\mathbf{K})$. In the example below, $\mathbf{V}'(\mathbf{K}) \models \top \to (\top \to \bot) \Longrightarrow \top \to \bot$, but

$$\mathbf{K} \not\models \frac{\top \to \bot}{\bot}\ .$$



A functional well-formed theory Γ is *faithful* if for all sets of constants C and all sequences of sentences $\forall\mathbf{x}(A_0 \to B_0), \forall\mathbf{x}(A_1 \to B_1), \ldots, \forall\mathbf{x}(A_n \to B_n)$, if

$$\Gamma\langle C\rangle \vdash \forall\mathbf{x}(A_n \to B_n) \wedge \cdots \wedge \forall\mathbf{x}(A_n \to B_n) \Longrightarrow \forall\mathbf{x}(A_0 \to B_0),$$

then

$$\Gamma\langle C\rangle \cup \{A_1 \Longrightarrow B_1, \ldots, A_n \Longrightarrow B_n\} \vdash A_0 \Longrightarrow B_0.$$

A theory Γ is *finitely strongly complete* with respect to a class $\mathcal{K}$ of models, if Γ is complete with respect to $\mathcal{K}$, and if, moreover, for all sequent theories $\Delta \supseteq \Gamma$ that are generated by adding finitely many sequents, there is a subclass of models of $\mathcal{K}$ such that Δ is complete with respect to the subclass.

Theorem 5.12 *Let Γ be a functional well-formed theory, C an infinite set of new constant symbols, and $\mathcal{K}$ be a class of rooted models with respect to which $\Gamma\langle C\rangle$ is finitely strongly complete. Then the following are equivalent.*

1. *Γ is faithful.*
2. *If $\mathbf{K} \in \mathcal{K}$ has irreflexive root α, and $\gamma \in \Gamma\langle C\rangle$ is of the form $\forall\mathbf{x}(A_1 \to B_1) \wedge \cdots \wedge \forall\mathbf{x}(A_n \to B_n) \Longrightarrow A_0 \to B_0$ with all $\forall\mathbf{x}(A_i \to B_i)$ and $A_0 \to B_0$ sentences, then $\mathbf{V}'_S(\mathbf{K}) \models \gamma$, for some S.*
3. *If $\mathbf{K} \in \mathcal{K}$ has irreflexive root α, and $\gamma = (A \Longrightarrow B) \in \Gamma\langle C\rangle$ where A and B are sentences with all quantifier variables of B only occurring inside implication subformulas, then $\mathbf{V}'_\varnothing(\mathbf{K}) \models \gamma$ or $\mathbf{V}'(\mathbf{K}) \models \gamma$.*

Proof: Obviously, the third item implies the second. Assume the second item. To derive the first item, let $\Gamma\langle C\rangle \vdash \forall\mathbf{x}(A_1 \to B_1) \wedge \cdots \wedge \forall\mathbf{x}(A_n \to B_n) \Longrightarrow \forall\mathbf{x}(A_0 \to B_0)$ with all $\forall\mathbf{x}(A_i \to B_i)$ sentences, and set $\Delta = \Gamma\langle C\rangle \cup \{A_1 \Longrightarrow B_1, \ldots, A_n \Longrightarrow B_n\}$. To prove that $\Delta \vdash A_0 \Longrightarrow B_0$, we may assume Δ to be consistent. Let $D \subseteq C$ be a finite subset including all new constant symbols that occur in all sentences $\forall\mathbf{x}(A_i \to B_i)$. Then $\Gamma\langle D\rangle \vdash \forall\mathbf{x}(A_1 \to B_1) \wedge \cdots \wedge \forall\mathbf{x}(A_n \to B_n) \Longrightarrow \forall\mathbf{x}(A_0 \to B_0) \Longrightarrow A_0 \to B_0 \Longrightarrow (A_0 \to B_0)^{\mathbf{x}}_{\mathbf{e}}$ with constant symbols $\mathbf{e}$ from $E = C \setminus D$. Let $\mathbf{K} \in \mathcal{K}$ be a model of Δ with root α. We want to show that $\mathbf{K} \models (A_0 \Longrightarrow B_0)^{\mathbf{x}}_{\mathbf{e}}$. If α is reflexive, then this is immediate from the definitions. So assume that α is irreflexive. Consider the model $\mathbf{V}'_S(\mathbf{K})$ of the condition, with new root $\alpha_0 \Vdash \forall\mathbf{x}(A_1 \to B_1) \wedge \cdots \wedge \forall\mathbf{x}(A_n \to B_n) \Longrightarrow (A_0 \to B_0)^{\mathbf{x}}_{\mathbf{e}}$. Then $\alpha_0 \Vdash \forall\mathbf{y}(A \to B)$ exactly when $\alpha \Vdash A \Longrightarrow B$, for all sentences $\forall\mathbf{y}(A \to B)$. So $\alpha \Vdash (A_0 \Longrightarrow B_0)^{\mathbf{x}}_{\mathbf{e}}$, and thus $\mathbf{K} \models (A_0 \Longrightarrow B_0)^{\mathbf{x}}_{\mathbf{e}}$. By finite strong completeness, $\Delta = \Gamma\langle D\rangle\langle E\rangle \cup \{A_1 \Longrightarrow B_1, \ldots, A_n \Longrightarrow B_n\} \vdash (A_0 \Longrightarrow B_0)^{\mathbf{x}}_{\mathbf{e}}$. So, by functional generalization, $\Gamma\langle D\rangle \cup \{A_1 \Longrightarrow B_1, \ldots, A_n \Longrightarrow B_n\} \vdash A_0 \Longrightarrow B_0$. And thus $\Delta \vdash A_0 \Longrightarrow B_0$.

Assume the first item. To derive the third item, let $\mathbf{K} \in \mathcal{K}$ have irreflexive root α, and let $\gamma \in \Gamma\langle C\rangle$ be of the required form. Write $\gamma = A \Longrightarrow B$. If $\mathbf{K} \not\models A$, then $\mathbf{V}'_S(\mathbf{K}) \models A \Longrightarrow B$ for all S; so we may assume that $\mathbf{K} \models A$. Additionally, we may assume that $\mathbf{V}'_\varnothing(\mathbf{K}) \not\models A \Longrightarrow B$. Let α_0 be the root of $\mathbf{V}'_\varnothing(\mathbf{K})$. Then $\alpha_0 \Vdash A$ and $\alpha_0 \nVdash B$. Up to provable equivalence A equals an expression $\exists\mathbf{y}(D_1 \vee \cdots \vee D_m)$ of D_i that are conjunctions of atoms and universal quantifications. So $\alpha_0 \Vdash D^{\mathbf{y}}_{\mathbf{c}}$ for some $D \in \{D_1, \ldots, D_m\}$ and with $\mathbf{c}$ from $C\alpha$. By transitivity, $\Gamma\langle C\rangle \vdash D^{\mathbf{y}}_{\mathbf{c}} \Longrightarrow B$. Since $\alpha_0 \nVdash p$ for all nontrivial atomic sentences p, $D^{\mathbf{y}}_{\mathbf{c}}$ must be BQC-equivalent to a conjunction $\forall\mathbf{x}(A_1 \to B_1) \wedge \cdots \wedge \forall\mathbf{x}(A_n \to B_n)$, so $\mathbf{K} \models A_i \Longrightarrow B_i$ for all i. So $\mathbf{V}'_S(\mathbf{K}) \models D^{\mathbf{y}}_{\mathbf{c}}$ for all S. Up to provable equivalence B equals a conjunction $E_1 \wedge \cdots \wedge E_m$ of E_i that are disjunctions of atoms and implications. Let $E \in \{E_1, \ldots, E_m\}$. It suffices to show that $\mathbf{V}'(\mathbf{K}) \models E$. We have $\mathbf{K} \models D^{\mathbf{y}}_{\mathbf{c}} \Longrightarrow E$ and, by transitivity, $\Gamma\langle C\rangle \vdash D^{\mathbf{y}}_{\mathbf{c}} \Longrightarrow E$. Now $\mathbf{V}'(\mathbf{K}) \models p$ exactly when $\mathbf{V}'(\mathbf{K}) \models \top \to p$, for all atomic sentences p. Replace all p

in E that are not inside implications by implications $\top \to p$, resulting in a disjunction called F. It suffices to show $\mathbf{V}'(\mathbf{K}) \models F$. Suppose not. Clearly, F is BQC-equivalent to a disjunction $(G_1 \to H_1) \vee \cdots \vee (G_k \to H_k)$, and $\Gamma\langle C\rangle \vdash D^{\mathbf{y}}_{\mathbf{c}} \Longrightarrow F$. If for some i we have $\alpha \nVdash G_i$ and $\alpha \Vdash G_i \to H_i$, then $\alpha \Vdash G_i \Longrightarrow H_i$, so $\mathbf{V}'(\mathbf{K}) \models F$. So we may assume there is $m \leq k$ such that $\alpha \Vdash G_i$ exactly when $i \leq m$, and $\alpha \nVdash G_i \to H_i$ for all $i > m$. Now $\Gamma\langle C\rangle$ proves

$$(\top \to \bigwedge_{i\leq m} G_i) \wedge D^{\mathbf{y}}_{\mathbf{c}} \Longrightarrow \bigvee_{i\leq m}(\top \to H_i) \vee \bigvee_{i>m}(\top \to (G_i \to H_i)) \Longrightarrow$$
$$\top \to (\textstyle\bigvee_{i\leq m} H_i \vee \bigvee_{i>m}(G_i \to H_i)).$$

By faithfulness,

$$\Gamma\langle C\rangle \cup \{\bigwedge_{i\leq m} G_i, A_1 \Longrightarrow B_1, \ldots, A_n \Longrightarrow B_n\} \vdash \bigvee_{i\leq m} H_i \vee \bigvee_{i>m}(G_i \to H_i).$$

So $\alpha \Vdash G_i \Longrightarrow H_i$ for some $i \leq m$. □

As a corollary we get:

Proposition 5.13 *Let Γ be a functional well-formed theory whose class of Kripke models is closed under the following transformation: if $\mathbf{K}$ is a rooted Kripke model of Γ with irreflexive root, then so is $\mathbf{V}'_S(\mathbf{K})$, for some S. Then Γ is faithful.*

There is a property stronger than faithfulness for which it would be nice to have a closure characterization similar to the one defining faithfulness. Unfortunately that turned out to be more cumbersome than expected. So, instead, we give a model-theoretic characterization. Let Γ be a functional well-formed theory over a language $\mathcal{L}$ with at least one constant symbol. So the set T of closed terms is nonempty. Set $S = T/\sim$, where $\sim$ is the equivalence relation on T defined by $s \sim t$ exactly when $\Gamma \vdash s = t$. Now for each set of models $\{\mathbf{K}_s\}_{s\in S}$ of Γ we can construct two new models, named $\mathbf{U}_r$ and $\mathbf{U}_i$, as follows. Both are formed by taking the disjoint union of the models $\mathbf{K}_s$, and then adding a new root α_0 with domain $D\alpha_0 = S$. In model $\mathbf{U}_r$ the node α_0 is reflexive; in model $\mathbf{U}_i$ the node α_0 is irreflexive. Let $\mathcal{K}$ be the class of rooted Kripke models of a functional well-formed theory Γ. We call Γ *reflexively rooted* if for each set of models $\{\mathbf{K}_s\}_{s\in S} \subseteq \mathcal{K}$ of Γ, the model $\mathbf{U}_r$ is also a model of Γ. Similarly, we call Γ *irreflexively rooted* if for each set of models $\{\mathbf{K}_s\}_{s\in S} \subseteq \mathcal{K}$ of Γ, the model $\mathbf{U}_i$ is also a model of Γ. The theory Γ is *fully rooted* if both $\mathbf{U}_r$ and $\mathbf{U}_i$ are models of Γ.

Recall that a theory Γ satisfies the *disjunction property* if $\Gamma \vdash A \vee B$ implies $\Gamma \vdash A$ or $\Gamma \vdash B$, for all sentences A and B. A theory Γ is said to satisfy *explicit definability* if $\Gamma \vdash \exists x A$ implies $\Gamma \vdash A^x_t$ for some term t, for all sentences $\exists x A$.

Proposition 5.14 *A functional well-formed theory which is reflexively rooted or irreflexively rooted is faithful, satisfies the disjunction property, and satisfies explicit definability. BQC is fully rooted; IQC is reflexively rooted; FQC is irreflexively rooted.*

Proof: All forms of rootedness obviously imply faithfulness, the disjunction property, and explicit definability. The rootedness of BQC, IQC, and FQC requires that the set of closed terms is nonempty. But we can make it that way by adding a set of new constant symbols, and then apply the Functional Generalization Proposition 4.12. The reflexive rootedness of IQC is well known. The full rootedness of BQC immediately follows from the completeness theorem. Let $\{\mathbf{K}_s\}_{s\in S}$ be a set of models of FQC. Then

$$\alpha \Vdash \frac{A \wedge (\top \to B) \Longrightarrow B}{A \Longrightarrow B}$$

for all $\alpha \succ \alpha_0$, so $\mathbf{U}_i \models (\top \to B) \to B \Longrightarrow \top \to B$. Apply Proposition 4.1. □

When we start with an empty collection of models $\{\mathbf{K}_s\}_{s\in S}$, then the derived rooted models $\mathbf{U}_r$ and $\mathbf{U}_i$ have single nodes and are models of CQC and $\top \to \bot$ respectively. As easy consequences we get the following proposition.

Proposition 5.15 *Let Γ be a functional well-formed theory. If Γ is reflexively rooted, then CQC $+ \Gamma$ is consistent, and it satisfies the same geometric sentences as Γ. If Γ is irreflexively rooted, then $(\top \to \bot) + \Gamma$ is consistent, and it satisfies the same geometric sentences as Γ.*

Proof: The rootedness immediately implies the consistency of CQC $+ \Gamma$ or $(\top \to \bot) + \Gamma$. Consider the case that CQC $+ \Gamma$ is consistent, and let A be a geometric sentence such that CQC $+ \Gamma \vdash A$. Let $\mathbf{K}$ be a model of Γ. Form the model $\mathbf{U}_r$ from $\mathbf{K}$, with new root α. The bottom node structure by itself is a model of CQC $+ \Gamma$. Since A is a geometric sentence, $\alpha \Vdash A$. So $\mathbf{K} \models A$. And thus $\Gamma \models A$. The proof for the case that $(\top \to \bot) + \Gamma$ is consistent is essentially the same. □

6 Basic Arithmetic Basic Arithmetic is the basic logic equivalent of Heyting Arithmetic over intuitionistic logic, and of Peano Arithmetic over classical logic. The nonlogical symbols are a constant symbol 0, a unary function symbol S for successor, and the binary function symbols $\cdot$ and $+$. Basic Arithmetic (BA) is axiomatizable by the axiom sequents

$$Sx = 0 \Longrightarrow \bot;$$
$$Sx = Sy \Longrightarrow x = y;$$
$$x + 0 = x;$$
$$x \cdot 0 = 0;$$
$$x + Sy = S(x + y);$$
$$x \cdot Sy = (x \cdot y) + x;$$

the rule schema of induction

$$\frac{A \Longrightarrow A^x_{Sx}}{A^x_0 \Longrightarrow A};$$

and the axiom schema of induction

$$\forall \mathbf{y}x(A \to A^x_{Sx}) \Longrightarrow \forall \mathbf{y}x(A^x_0 \to A).$$

This completes the axiomatization of BA. It is an easy exercise to prove Proposition 6.1.

Proposition 6.1 *BA is functional and well-formed.*

Faithfulness, the disjunction property, and explicit definability are implied by the following preservation construction.

Proposition 6.2 *BA is fully rooted. HA = IQC + BA is reflexively rooted. FQC + BA is irreflexively rooted.*

Proof: Let $\mathbf{U} = \mathbf{U}_r$ or $\mathbf{U} = \mathbf{U}_i$ be constructed from a set of models $\{\mathbf{K}_s\}_{s \in S}$ of BA. We must show that $\alpha_0 \Vdash \gamma$ for all sequents of the definition of BA, and $\alpha_0 \Vdash R$, where R is an instance of the rule schema of induction. Only the induction schemas are nontrivial. Suppose $\alpha_0 \Vdash A \Longrightarrow A^x_{Sx}$. Then certainly $\alpha \Vdash A^x_0 \Longrightarrow A$ for all $\alpha \neq \alpha_0$. Let $\mathbf{y}$ include all free variables of A except x, and assume $\alpha_0 \Vdash A^{\mathbf{y},x}_{\mathbf{d},0}$ for some $\mathbf{d} \in \omega^n$. But $\alpha_0 \Vdash A^{\mathbf{y},x}_{\mathbf{d},n} \Longrightarrow A^{\mathbf{y},x}_{\mathbf{d},Sn}$ for all $n \in \omega$. So $\alpha_0 \Vdash A^{\mathbf{y},x}_{\mathbf{d},n}$ for all $n \in \omega$. So $\alpha_0 \Vdash A^x_0 \Longrightarrow A$. Finally, suppose $\alpha_0 \Vdash \forall \mathbf{y} x(A \rightarrow A^x_{Sx})$. If α_0 is reflexive, then apply the rule case above to conclude that $\alpha_0 \Vdash \forall \mathbf{y} x(A^x_0 \rightarrow A)$. Otherwise, if α_0 is irreflexive, use that all $\mathbf{K}_s$ are models of BA to conclude that $\alpha \Vdash A^x_0 \Longrightarrow A$, for all $\alpha \succ \alpha_0$. And thus $\alpha_0 \Vdash \forall \mathbf{y} x(A^x_0 \rightarrow A)$. The cases for HA and FQC + BA now easily follow with Proposition 5.14. □

Proposition 6.2 implies that $(\top \rightarrow \bot) + \mathrm{BA}$ is consistent, hence FQC + BA is consistent.

Corollary 6.3 *BA and FQC + BA are undecidable.*

Proof: By Matijasevič's undecidability theorem for Diophantine equations, the fragment of existential sentences derivable from PA is undecidable. But by Propositions 5.15 and 6.2, the theories BA and FQC + BA derive exactly the same existential sentences. □

This corollary naturally extends to other arithmetics like $(\top \rightarrow \bot) + \mathrm{BA}$.

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