Kolmogorov Translations into Basic Logic

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1 Intuitionistic Logic

There are, from a historical perspective, three major schools of constructive mathematics: Brouwer, Markov, and Bishop.

Heyting's intuitionistic predicate logic IQC of 1927–30 represents very general regularities observed in language used in constructive mathematical proofs. At least until the middle 1930s some criticized IQC as too strong for constructive mathematics. After 50 years IQC became treated as *the* standard (two reasons).

The intended meaning of the logical constants is clarified through Heyting's 1928–34 proof interpretation. Still criticized (one main reason).

2 Constructive Logic

Basic logic BQC is *the* constructive logic. The proposition logical part is due to Albert Visser, 1980–81.

A justification is based on a proof interpretation with triples (A, p, B) instead of pairs (p, B) as by Heyting or Kreisel (the idea of triples is not completely new). The limitations of BQC are based on weak counterexamples (similar to ones for IQC).

BQC is a subsystem of IQC with weakened modus ponens. No longer $\neg A \equiv \neg\neg\neg A$, but still $\neg\neg A \equiv \neg\neg\neg\neg A$.

BQC is complete for the class of transitive Kripke models.

IQC equals BQC plus $\top \to A \vdash A$.

3 Kolmogorov 1925

A significant partial version of IQC is due to Kolmogorov (1925).

Classical logic CQC is definable by adding $\neg\neg A \vdash A$ to IQC. Define translation $A \mapsto A^k$ by replacing all subformulas B by $\neg\neg B$ (defined by induction on the complexity of B). Kuroda's 1951 translation is equivalent.

$$\text{CQC} \vdash A \text{ if and only if IQC} \vdash A^k$$

So IQC has expressive strength.

4 BQC **Language Necessities**

Instead of $A \to B$ and $\forall x A$, the language of BQC has $\forall \mathbf{x}(A \to B)$, where $\mathbf{x}$ equals $x_1, \ldots, x_n$ with $n \geq 0$.

Former $\forall x A$ 're-appears' as $\forall x(\top \to A)$

New $\forall xy(A \to B)$ avoids problematic nested $\forall x \forall y$

$A \to B$ is now definable as $\forall(A \to B)$

Over BQC we still define $\neg A$ by $A \to \bot$.

5 A new Kolmogorov Translation

Classical logic CQC is definable by adding $\neg\neg A \vdash A$ to BQC. Define translation $A \mapsto A^k$ by replacing 'all' subformulas B by $\neg\neg B$ as follows:

$$\begin{aligned}
P^k &:= \neg\neg P \quad \text{for atoms } P \\
(A \wedge B)^k &:= \neg\neg(A^k \wedge B^k) \\
(A \vee B)^k &:= \neg\neg(A^k \vee B^k) \\
(\exists x A)^k &:= \neg\neg(\exists x\, A^k) \\
(\forall \mathbf{x}(A \to B))^k &:= \neg\neg(\forall \mathbf{x}\, \neg\neg(A^k \to B^k))
\end{aligned}$$

We have

$$(\forall \mathbf{x}(A \to B))^k \equiv \neg\neg(\forall \mathbf{x}(A^k \to B^k))$$

There is an equivalent translation based on Kuroda's 1951 version. A main theorem:

$$\text{CQC} \vdash A \text{ if and only if BQC} \vdash A^k$$

So BQC has expressive strength.