One Hundred Years of Logic for Constructive Mathematics

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An incomplete understanding of what is constructive mathematics suffices to uniquely determine constructive logic. It is not intuitionistic logic.

1 Classical versus Constructive

With some simplification:
Modern classical mathematics was not completely formed until the 1930s (Hilbert, Gödel, and others).
Constructive mathematics existed before 1910 and still is not completely formed (Brouwer, Markov, Bishop, and others).

Classical (predicate) logic was completely settled by 1930. It includes logical claims like A or not A (written $A \vee \neg A$). Boolean algebras are associated with part of it.

Intuitionistic (predicate) logic was defined by 1928, and was completely settled by the late 1950s ($A \vee \neg A$ is not included). Topological spaces are associated with part of it. We establish that intuitionistic logic is not constructive logic. We offer the alternative correct version.

2 Constructivism by Example

Let A be the statement (Goldbach Conjecture): All even integers bigger than 2 are sums of two primes.

Statement A may be true solely because it is not false while there exists no (classical) proof.

Classical mathematics claims $A \vee \neg A$.

Constructive mathematics and intuitionistic mathematics don't claim $A \vee \neg A$ because constructively one would be required to present either a proof of A or a proof of $\neg A$.

Better than A and $\neg A$: Let $B(n)$ be the (computable!) statement: Even integer $n \geq 4$ is the sum of two primes less than n. Replace A by $\forall n B(n)$. Replace $\neg A$ by $\exists n \neg B(n)$.

Constructive mathematics and intuitionistic mathematics expect that a proof of $\exists n \neg B(n)$ yields a construction of n for which $\neg B(n)$ holds.

3 Logic as Practiced

Based on Heyting (1978): As a study of regularities in language, logic is an experimental science in need of mathematical notions. It belongs to applied mathematics.

Brouwer (1923): In intuitionistic logic, $\neg A$ is equivalent to $\neg\neg\neg A$.

Kolmogorov (1925) introduces a significant part of intuitionistic logic.

Heyting (1928) introduces intuitionistic logic. Use logical constants $\top$ (true), $\bot$ (false), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies), $\exists$ (exists), and $\forall$ (for all). Negation $\neg A$ 'is' $A \rightarrow \bot$. Some (modern versions of) derivation rules as illustrations:

$$\frac{D, A \vdash C \quad D, B \vdash C}{D,\, A \vee B \vdash C}$$

and

$$\frac{D, A \vdash B \rightarrow C}{D, A, B \vdash C}$$

4 Logic and Meaning

Based on Heyting (1978): From the point of view of the intended meaning, logic expresses very general mathematical theorems.

There is no generally accepted justification of intuitionistic logic as *the* constructive logic. Heyting's so-called proof interpretation of 1934 is still the basis of most attempts at a justification.

Examples of proof interpretations of a few of the logical constants:

> A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B (plus the stipulation that we want to regard the proof presented as evidence for $A \vee B$).
>
> A proof of $A \to B$ is a construction which permits us to transform any proof of A into a proof of B.

There is no generally accepted process by which to convert the proof interpretation into a justification of intuitionistic logic.

5 Intuitionistic Logic as Standard

After about 50 years intuitionistic logic became treated as *the* constructive logic.

The maximum principle: Models (classical) of constructivist scenarios imply that extensions of intuitionistic logic are not constructive. For example Kripke models, pre-orders (reflexive and transitive) of classical structures with morphisms, present structures with future possible scenarios. Unsatisfactory but convincing.

The minimum principle: We skip some sophisticated arguments that attempt to justify all rules of intuitionistic logic. We present only practical arguments:

First, intuitionistic logic with its (Kripke and other) models has shown to be very useful, including for classical mathematics.

Second, for many years no viable useful alternative has been recognized. A weakening of intuitionistic logic may instead result in too weak a system to be seriously useful.

6 The Critics

Is intuitionistic logic constructive?

Many early objections involved the use of false $\bot$ and negation $\neg A$. Gödel objected that the broad use of $\forall n A(n)$ in intuitionistic arithmetic goes beyond computability.

Gödel (1938) in a private note suggested that the binary objects (p, B) of Heyting's proof interpretation (proof p of statement B) should be replaced by triples (A, p, B) of proof p of B from assumption A.

Main problem: The meaning of implication $A \to B$.

Markov (1960s–1970s) struggled to develop a large "stepwise" semantic system in order to achieve a satisfactory theory of implication.

Bishop (1967–1970), statements $(A \to B) \to C$ have a less immediate meaning than A, B, and C. The numerical meaning of implication is a priori unclear.

Dummett (1977–2000), we must, in some sense, be able to survey or grasp some totality of constructions which will include all possible proofs of a given statement.

7 New: Constructions and Proofs

Our justification is based on a proof interpretation with triples (A, p, B) (before becoming aware of Gödel's 1938 private note). There is an immediate conversion of proof interpretation into constructive logic (currently named Basic Logic).

Examples:

If we *have* proofs (A, p, C) and (B, q, C), then there is a proof which we name $(A \vee B, [p, q], C)$, and which we construct in a uniform way in terms of p and q. So

$$\frac{D, A \vdash C \quad D, B \vdash C}{D,\ A \vee B \vdash C}$$

Assume proofs (A, x, B) and (B, y, C) without specifying x and y any further. Construct composition proof (A, yx, C) in the hypothetical sense of constructivists. So

$$D,\ (A \to B),\ (B \to C) \vdash A \to C$$

NOT $\frac{D, A \vdash B \to C}{D, A, B \vdash C}$.

Basic Logic is *the* constructive logic. The part without $\exists$ and $\forall$ is due to Albert Visser, 1980–1981.

8 Boundaries of Basic Logic

Basic Logic is *the* constructive logic.

The minimum principle: The proof interpretation justifies all rules of Basic Logic.

The maximum principle: Models (classical) of constructivist scenarios imply that extensions of Basic Logic are not constructive. Kripke models on transitive sets (not always reflexive) of classical structures with morphisms, present structures with future possible scenarios. Unsatisfactory but convincing. The larger class of Kripke models allows for the simulation of a growing accumulation of proofs.

9 Logic Comparisons

Basic Logic is a subsystem of intuitionistic logic. We no longer have Brouwer's 1923 equivalence of $\neg A$ and $\neg\neg\neg A$. We still have $\neg\neg A$ and $\neg\neg\neg\neg A$ equivalent.

Intuitionistic logic can be obtained from Basic Logic by the addition of axiom $\top \to A \vdash A$.

Classical logic can be obtained from Basic Logic by the addition of axiom $\neg\neg A \vdash A$.

Is Basic Logic too weak? The embeddings by Kolmogorov (1925) and Kuroda (1951) of classical logic into Basic Logic work as they do into intuitionistic logic.

More to come.