# Kolmogorov and Kuroda Translations into Basic Predicate Logic

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**Abstract**

Kolmogorov established the principle of the double negation translation by which to embed Classical Predicate Logic CQC into Intuitionistic Predicate Logic IQC. We show that the obvious generalizations to the Basic Predicate Logic of [ArRu20] and to BQC of [Ru98], a proper subsystem of IQC, go through as well. The obvious generalizations of Kuroda's embedding are shown to be equivalent to the Kolmogorov variant. In our proofs novel nontrivial techniques are needed to overcome the absence of full modus ponens in Basic Predicate Logic.

In [ArRu20] we argued that IQC is not the logic of constructive mathematics. Our doubts were far from new. New was that we put forward an alternative, BQC. One concern is that BQC is too weak for serious mathematics, or even trivial. This paper is one step to alleviate such concerns.

## 1 Introduction

There is a common misconception that logic without modus ponens $A \wedge (A \to B) \vdash B$ is too weak for the derivation of much mathematics of substance. One motivation for this paper is to show that Basic Predicate Logic BQC of [Ru98] has much logical strength despite the absence of full modus ponens. Our straightforward generalizations of the Kolmogorov and Kuroda embeddings of [Ko25] and [Ku51] are quick ways to establish this. We embed CQC into BQC, a proper subsystem of IQC.

The Kolmogorov embedding from CQC into IQC puts double negations in front of all subformulas. We show that this robust embedding even works for the Basic Predicate Logic of [ArRu20], so also works for BQC. Instead of replacing subformulas $A$ by $\neg\neg A$, we employ a propositional letter $a$, and replace subformulas $A$ by $(A \to a) \to a$. This extends the proposition logical translation implied by [AAR16] to predicate logic. Parameter $a$ allows for a slightly more detailed analysis of the translations. The Kuroda translation turns out to be a more parsimonious equivalent of the Kolmogorov translation. Kuroda's original translation of [Ku51], from CQC into IQC, is an extension of Glivenko's proposition logical translation of [Gl29].

Generalizations along the lines of the Gödel and Gentzen translations are slightly more restricted versions of the translations in this paper. We discuss them in another paper when we consider aspects of Basic Arithmetic and embeddings of Peano Arithmetic into Basic Arithmetic.

Although the Kolmogorov and Kuroda translations are equivalent, generalizations need not be. Brown and Rizkallah in [BrRi14] prefer to extend the intuitionistic Kuroda translation to variants of simple intuitionistic type theories. Van den Berg in [vdB19] makes no mention of Glivenko or Kolmogorov. The intuitionistic Kuroda translation is generalized to special $j$ translations or nuclei.

# 2 Basic Logic Axioms and Rules

The Basic Predicate Logic axiom and rule system BQC-E below is derived from and equivalent to the one in the paper [ArRu20] except for an extension with function symbols and equality. At the end of this section we clarify its connection with the Basic Predicate Logic BQC of [Ru98]. Here we are not trying to show that BQC is *the* constructive predicate logic. So we use a simpler notation.

We call the Basic Predicate Logic below BQC-E so as to distinguish it from the BQC of [Ru98]. Its language $\mathcal{L}$-E still needs universal implication $\forall \mathbf{x}(A \to B)$, a notation which combines implication with universal quantification over finite lists of variables $\mathbf{x}$. Implication $A \to B$ is short for the special case $\forall(A \to B)$. This convention creates the minor difficulty that $A \to B$ cannot be named quantifier free without some confusion. We rescue this by calling formulas including $A \to B$ quantified variable free (assuming $A$ and $B$ are) instead of quantifier free. Proposition logical formulas are the ones without quantified variables. Proposition logical combinations of formulas are formed by combining them using only quantified variable free extra formulas and $\wedge$, $\vee$, and $\to$.

We include the existence predicate $\mathrm{E}\,x$ of [Sc79] for the following reason. In classical logic, models with empty domain trivially reduce to propositional logic, so models with empty domain are unnecessary for Classical Predicate Logic CQC. Not so in the constructive case. Even in intuitionism one can imagine interesting structures with the property that 'the domain not not has an element,' which in constructive mathematics differs from 'the domain has an element'. Kripke models are one common way to establish such differences. See [Dr79] or [TrvD88].

We write entailment $\vec{A} \vdash B$ where $\vec{A} = (A_1, A_2, \ldots, A_n)$ is a list of formulas, with intended meaning that formula $B$ is derivable from the formulas in list $\vec{A}$. We write $\vdash B$ when $n = 0$. Write $\vec{A}, A_{n+1}$ as alternate for $(A_1, A_2, \ldots A_{n+1})$ or $A_0, \vec{A}$ for $(A_0, A_1, \ldots, A_n)$. We ignore formula parentheses when it improves readability.

We freely reorder, or add and remove duplicate entries, in lists $\vec{A}$. For entailment $\vdash$ we have axioms and rules

$$A \vdash A \quad \text{and} \quad \frac{\vec{D} \vdash B}{\vec{D}, A \vdash B} \quad \text{and} \quad \frac{\vec{D}, A \vdash B \quad \vec{D}, B \vdash C}{\vec{D}, A \vdash C}$$

We have predicates $P(x_1, x_2, \ldots, x_m)$ of arities $m \geq 0$. There is a special predicate $x = y$ for the equality relation, and a special predicate $\mathrm{E}\,x$ for existence. We have partial function symbols $f(x_1, x_2, \ldots, x_m)$ of arities $m \geq 0$. We don't require them to be total. In particular, constant symbols are partial function symbols of arity 0. For example we allow the possibility of constant symbols $c$ for which we have $\neg\neg\,\mathrm{E}\,c$ but not $\mathrm{E}\,c$. We have a constant symbol $\ell$ for undefined element. Terms are defined in the usual way as compositions using function symbols and variables.

For convenience we may write $\mathbf{x}$ for lists $x_1, x_2, \ldots, x_m$ of variables of finite length $m \geq 0$. We write $\mathbf{xy}$ for concatenated lists $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ or $\mathbf{x}y$ for $x_1, x_2, \ldots, x_m, y$. For lists of terms $t_1, t_2, \ldots, t_k$ we use similar conventions. We write $P(\mathbf{x})$ for atoms $P(x_1, x_2, \ldots, x_m)$ and $t(\mathbf{x})$ for terms $t(x_1, x_2, \ldots, x_m)$.

We have the usual $\wedge$, $\vee$, $\top$, $\bot$, and $\exists$ of the intuitionistic predicate logic language, where $\top$ and $\bot$ are both predicates and logical constants. Instead of implication and universal quantification we have universal implication construction $\forall\mathbf{x}(A \to B)$, where list $\mathbf{x}$ is allowed to be empty. We ignore the order of the variables in $\mathbf{x}$, or duplications among them. We write $A \to B$ as short for $\forall(A \to B)$. Negation $\neg A$ is defined by $A \to \bot$, and bi-implication $\forall\mathbf{x}(A \leftrightarrow B)$ is defined by $\forall\mathbf{x}(A \to B) \wedge \forall\mathbf{x}(B \to A)$.

With $\wedge$ we have rules

$$\frac{\vec{D}, A, B \ \vdash\ C}{\vec{D},\, A \wedge B \ \vdash\ C} \quad \text{and} \quad \frac{\vec{D},\, A \wedge B \ \vdash\ C}{\vec{D}, A, B \ \vdash\ C}$$

and

$$\frac{\vec{D} \ \vdash\ A \wedge B}{\vec{D} \ \vdash\ A} \quad \text{and} \quad \frac{\vec{D} \ \vdash\ A \wedge B}{\vec{D} \ \vdash\ B} \quad \text{and} \quad \frac{\vec{D} \ \vdash\ A \quad \vec{D} \ \vdash\ B}{\vec{D} \ \vdash\ A \wedge B}$$

For $\top$ we have axiom

$$\vdash \top$$

For $\vee$ we have rules

$$\frac{\vec{D},\, A \vee B \ \vdash\ C}{\vec{D}, A \ \vdash\ C} \quad \text{and} \quad \frac{\vec{D},\, A \vee B \ \vdash\ C}{\vec{D}, B \ \vdash\ C} \quad \text{and} \quad \frac{\vec{D}, A \ \vdash\ C \quad \vec{D}, B \ \vdash\ C}{\vec{D},\, A \vee B \ \vdash\ C}$$

Lists $\vec{D}$ of additional assumptions are key in proving distributivity. Straightforward proofs yield $A, B \ \vdash\ (A \wedge B) \vee (A \wedge C)$ and $A, C \ \vdash\ (A \wedge B) \vee (A \wedge C)$, so also

$$A,\, B \vee C \ \vdash\ (A \wedge B) \vee (A \wedge C)$$

Thus we prove

$$A \wedge (B \vee C) \ \vdash\ (A \wedge B) \vee (A \wedge C)$$

The symbol $\bot$ can be left out for theories that don't have an acceptable candidate for 'false'. For the theory of arithmetic an atom like $1 = 0$ can perform the role. If we include $\bot$, then we have axioms

$$\bot \vdash A$$

We write $\mathrm{E}\,\mathbf{x}$ as short for $\mathrm{E}\,x_1 \wedge \mathrm{E}\,x_2 \wedge \ldots \wedge \mathrm{E}\,x_m$.
Predicates $P$ are strict, so we have axioms

$$P(\mathbf{x}) \ \vdash\ \mathrm{E}\,\mathbf{x}$$

For the equality predicate we have axioms

$$\mathrm{E}\,x \ \vdash\ x = x \quad \text{and} \quad A \wedge x = y \ \vdash\ A[x/y] \text{ for atoms } A$$

Functions may be partial, but they are strict. So we have axioms

$$\mathrm{E}\,f(\mathbf{x}) \ \vdash\ \mathrm{E}\,\mathbf{x}$$

For the constant symbol for an undefined element we have axiom

$$\mathrm{E}\,\ell \ \vdash\ \bot \quad \text{(or schema } \mathrm{E}\,\ell \ \vdash\ A \text{ if } \bot \text{ is left out)}$$

So $\mathrm{E}\,\ell$ can play the role of 'false'.

We have substitution rule

$$\frac{\vec{D} \ \vdash\ B}{\vec{D}[x/t] \ \vdash\ B[x/t]} \quad \text{no variable of term } t \text{ becomes bound}$$

For $\exists$ we have rules

$$\frac{\vec{D},\, A \wedge \mathrm{E}\, x \ \vdash\ B}{\vec{D},\, \exists x A \ \vdash\ B}\ x \text{ not free in } B, \vec{D} \qquad \text{and} \qquad \frac{\vec{D},\, \exists x A \ \vdash\ B}{\vec{D},\, A \wedge \mathrm{E}\, x \ \vdash\ B}$$

Lists $\vec{D}$ of additional assumptions are key in proving existential distributivity. A straightforward proof yields $A,\, B \wedge \mathrm{E}\, x \ \vdash\ \exists x(A \wedge B)$. So if $x$ is not free in $A$, then

$$A,\, \exists x B \ \vdash\ \exists x(A \wedge B)$$

Thus we prove

$$A \wedge \exists x B \ \vdash\ \exists x(A \wedge B) \qquad x \text{ not free in } A$$

We write $\exists \mathbf{x} A$ as short for $\exists x_1 \exists x_2 \ldots \exists x_n A$.

For universal implication we have rule

$$\frac{\vec{D},\, A \wedge \mathrm{E}\, \mathbf{x} \ \vdash\ B}{\vec{D} \ \vdash\ \forall \mathbf{x}(A \to B)} \qquad \text{variables } \mathbf{x} \text{ not free in } \vec{D}$$

(so $\vec{D}, A \ \vdash\ B$ implies $\vec{D} \ \vdash\ A \to B$), and axioms

$$\forall \mathbf{x}(A \to B) \ \vdash\ \forall \mathbf{x} y(A \to B) \qquad y \text{ not free to the left of the entailment}$$
$$\forall \mathbf{x} y(A \to B) \ \vdash\ \forall \mathbf{x}((A \wedge \mathrm{E}\, y) \to B)$$

Finally the 'formalization' axioms

$$\forall \mathbf{x}(A \to B) \wedge \forall \mathbf{x}(B \to C) \ \vdash\ \forall \mathbf{x}(A \to C)$$
$$\forall \mathbf{x}(A \to B) \wedge \forall \mathbf{x}(A \to C) \ \vdash\ \forall \mathbf{x}(A \to (B \wedge C))$$
$$\forall \mathbf{x}(B \to A) \wedge \forall \mathbf{x}(C \to A) \ \vdash\ \forall \mathbf{x}((B \vee C) \to A)$$
$$\forall \mathbf{x} y(A \to B) \ \vdash\ \forall \mathbf{x}((A[y/t] \wedge \mathrm{E}\, t) \to B[y/t]) \qquad \text{no variable of term } t \text{ becomes bound in } A \text{ or } B$$
$$\forall \mathbf{x} y(A \to B) \ \vdash\ \forall \mathbf{x}(\exists y A \to B) \qquad y \text{ not free in } B$$

We freely rename variables bound by the quantifiers $\exists$ and $\forall$, with the usual restrictions that with substitution the new variables do not become bound by other quantifiers, and unchanged variables do not become bound by the new variables attached to the quantifier.

This completes an axiomatization of BQC-E of the Basic Predicate Logic of [ArRu20] extended with function symbols and equality. Define Intuitionistic Predicate Logic IQC-E by the addition of schema $\top \to A \ \vdash\ A$, which allows one to derive modus ponens. Define Classical Predicate Logic CQC-E by adding schemas $\top \to A \ \vdash\ A$ plus Excluded Middle $\vdash\ A \vee \neg A$ or, alternatively, by adding only the schema of double negation elimination $\neg\neg A \ \vdash\ A$.

BQC-E embeds into the Basic Predicate Logic BQC of [Ru98] by adding a new predicate $Fx$ to the language of BQC, and relativize BQC-E quantifiers to $F$ while sending E to $F$. The presence of $\ell$ makes that over the BQC language with extra predicate $Fx$ the overall domain has an element, as is required. Equality of BQC-E becomes a congruence on $\{x \mid Fx\}$. Over BQC we replace partial $n$-ary function symbols by $(n+1)$-ary predicates.

We can define BQC as a theory over BQC-E after restricting its language $\mathcal{L}$-E to sub-language $\mathcal{L}$ as follows. Remove E and $\ell$ from $\mathcal{L}$-E, and axiom $\mathrm{E}\,\ell \vdash \bot$ from BQC-E. In the remaining axioms of BQC-E replace all occurrences of $\mathrm{E}\, t$ by $\top$. The result is a new BQC over $\mathcal{L}$, equivalent to the old BQC of [Ru98]. Expressions $\vec{D} \vdash B$ of this new BQC correspond with expressions $\bigwedge \vec{D} \ \Rightarrow\ B$ of the old BQC of 1998.

We call a theory over language $\mathcal{L}$-E $\ell$-free if we only consider derivations that don't mention $\ell$. This is our pragmatic way to define theories over the sub-language of $\mathcal{L}$-E formed by removing $\ell$ and axiom $\mathrm{E}\,\ell \vdash \bot$.

**Proposition 2.1.** *Let $\vec{D}_0 \vdash B_0, \ldots, \vec{D}_n \vdash B_n$ be a list of entailments over language $\mathcal{L}$ of* BQC*. For all $i$ let $\mathbf{x}_i$ be a list of the free variables of $\vec{D}_i \vdash B_i$. Let $\Gamma$ be the $\ell$-free theory* (BQC -*E*) $\cup \{\vdash \exists x \top\} \cup \{\mathrm{E}\,\mathbf{x} \vdash \mathrm{E}\, f(\mathbf{x})\}_f$*. Then*

> *There is a* BQC *derivation of $\vec{D}_0 \vdash B_0$ from $\{\vec{D}_i \vdash B_i\}_{1\leq i\leq n}$*
> *if and only if*
> *There is a $\Gamma$ derivation of $\vec{D}_0, \mathrm{E}\,\mathbf{x}_0 \vdash B_0$ from $\{\vec{D}_i, \mathrm{E}\,\mathbf{x}_i \vdash B_i\}_{1\leq i\leq n}$*

*Proof.* Suppose the first claim. We convert each step in a derivation tree over BQC into a short derivation over $\Gamma$. Most rules almost immediately convert into rules over $\Gamma$. One concern is the possibility that the conclusion entailment has too many additional free variables. Here is its solution: Over language $\mathcal{L}$-E suppose $\vec{D}_0, \mathrm{E}\,\mathbf{x}_0\mathbf{y} \ \vdash \ B_0$ follows from a list $\vec{D}_i, \mathrm{E}\,\mathbf{x}_i \ \vdash \ B_i$. If $\mathbf{x}_0$ contains some variable $x$, then with substitution of $x$ for all variables in $\mathbf{y}$ we get $\vec{D}_0, \mathrm{E}\,\mathbf{x}_0 \ \vdash \ B_0$. If $\mathbf{x}_0$ is empty, then $\vec{D}_0, B_0$ is a list of sentences, so $\vec{D}_0, \mathrm{E}\,x \ \vdash \ B_0$ for some auxiliary variable $x$, so also $\vec{D}_0, \exists x\top \ \vdash \ B_0$. Another concern is that with substitution we may introduce extra expressions $\mathrm{E}\,t(\mathbf{x})$ in the entailment. Already $\ell$-free BQC-E proves $\mathrm{E}\,t(\mathbf{x}) \vdash \mathrm{E}\,\mathbf{x}$. Theory $\Gamma$ provides the reverse $\mathrm{E}\,\mathbf{x} \vdash \mathrm{E}\,t(\mathbf{x})$.

Suppose the second claim. Because of the free permission of $\vec{D}$ on the left, we can replace in all axioms and rules of $\Gamma$, for all $i$ entailment $\vec{D}_i \vdash B_i$ by $\vec{D}_i, \mathrm{E}\,\mathbf{x}_i \vdash B_i$. For example $\vec{D}, \mathrm{E}\,\mathbf{y} \ \vdash \ \forall\mathbf{x}(A \to B)$ follows from $\vec{D}, A, \mathrm{E}\,\mathbf{x}\mathbf{y} \ \vdash \ B$. Variables among $\mathbf{y}$ that don't occur free elsewhere in the entailment can be removed using axiom $\vdash \exists x\top$ of $\Gamma$. With the further axiom set $\{\mathrm{E}\,\mathbf{x} \vdash \mathrm{E}\,f(\mathbf{x})\}_f$ we replace all $\vec{D}_i, B_i$ by $\vec{D}'_i, B'_i$ without occurrences of $\mathrm{E}\,t$. The resulting rule after removal of all $\mathrm{E}\,\mathbf{x}_i$ is $\vec{D}'_0 \vdash B'_0$ derivable from the $\vec{D}'_i \vdash B'_i$ over BQC. □

BQC of [Ru98] embeds conservatively into $\ell$-free theory (BQC-E)$\cup\{\vdash \exists x\top\}\cup\{\mathrm{E}\,\mathbf{x} \vdash \mathrm{E}\,f(\mathbf{x})\}_f$ by translation (writing $A \vdash B$ instead of $A \Rightarrow B$)

$$A \vdash B \quad \mapsto \quad A \wedge \mathrm{E}\,\mathbf{x} \vdash B \quad \text{where } \mathbf{x} \text{ is all free variables of } A \wedge B$$

Recall that intuitionistic predicate logic IQC is sound and complete for reflexive transitive Kripke models. Similarly, BQC is sound and complete for transitive Kripke models, see [Ru98]. There is no Kripke model completeness theorem for BQC-E.

In Sections 3 and 4 we essentially never use $\ell$. So their results immediately translate to BQC.

# 3 Basic Logic Properties

This section lists BQC-E properties many of whom are used in Sections 4 and 5. The list also illustrates how one can derive certain kinds of formulas. We don't attempt to have a complete list.

We write $A \equiv B$ as short for $A \vdash B$ plus $B \vdash A$. So $\equiv$ is the congruence associated with pre-order $\vdash$. We add a propositional letter $a$ to the predicate logic language for later use in our generalized Kolmogorov and Kuroda translations. Following [AAR16] we write $A^a$ as short for $A \to a$ and $A^{aa}$ as short for $(A \to a) \to a$, and so on. On occasion we may do the same for $A^B$ or $A^{BB}$. For universal implication in general we may write $\forall\mathbf{x}\, A^B$ or $\forall\mathbf{x}(A^B)$ as alternates for $\forall\mathbf{x}(A \to B)$.

We write $A[p]$ for formula contexts, where $p$ is a proposition letter to hold substitution places. Substitution $A[B]$ is the result of replacing all occurrences of $p$ by formula $B$. We restrict ourselves to substitution cases where no free variable of $B$ becomes bound after substitution for $p$ in $A[p]$, if necessary after renaming bound variables of $A[p]$.

**Proposition 3.1** (Substitution)**.** *Let $D[p]$ be a context with all bound variables distinct from the free variables of $A$, $B$, and $C$. Then $A \wedge B \ \vdash \ C$ plus $A \wedge C \ \vdash \ B$ implies*

$A \wedge D[B] \ \vdash \ D[C]$. *In particular we have* $B \equiv C$ *implies* $D[B] \equiv D[C]$*, and we have relative meet substitution* $A \wedge D[A \wedge B] \ \equiv \ A \wedge D[B]$*.*

*Proof.* By induction on the complexity of $D[p]$, see the proof of [Ru98, Proposition 4.3]. There is a minor difference with the induction step for existential quantification because of our inclusion of the existence predicate $\mathrm{E}\,x$. Suppose $D[p]$ equals $\exists x E[p]$ with $x$ not free in $A$, $B$ or $C$, possibly after renaming this bound variable. By induction we have $A \wedge E[B] \ \vdash \ E[C]$, so also $A \wedge E[B] \wedge \mathrm{E}\,x \ \vdash \ E[C] \wedge \mathrm{E}\,x \ \vdash \exists x E[C]$. Thus $A \wedge \exists x E[B] \ \vdash \ \exists x(A \wedge E[B]) \vdash \exists x E[C]$. □

We define positive and negative occurrences of $p$ in $A[p]$ in the usual way.

**Proposition 3.2.** *Let* $D[p]$ *be a context with all bound variables distinct from the free variables of* $A$ *and* $B$*. If* $p$ *occurs only in positive places in* $D[p]$*, then* $D[A \wedge B] \ \vdash \ D[A]$*. If* $p$ *occurs only in negative places in* $D[p]$*, then* $D[A] \ \vdash \ D[A \wedge B]$*.*

*Proof.* Both statements are proved simultaneously by induction on the complexity of $D[p]$ by a straightforward variation on the proof of [Ru98, Proposition 4.3]. □

A *sequent theory* is a theory axiomatizable by a set of entailments $A \vdash B$. For example IQC-E and CQC-E are sequent theories. Following [Ru98, page 33], a sequent theory $\Gamma$ is *well-formed* if for all finite lists of sentences $\forall \mathbf{x}(A_0 \to B_0)$, $\forall \mathbf{x}(A_1 \to B_1)$, $\ldots$, $\forall \mathbf{x}(A_n \to B_n)$ and formula $A$ where no free variable of $\mathbf{x}$ occurs in $A$, if

$$\Gamma \cup \{A_1 \wedge \mathrm{E}\,\mathbf{x} \ \vdash \ B_1, \ldots, A_n \wedge \mathrm{E}\,\mathbf{x} \ \vdash \ B_n\} \text{ proves } A_0 \wedge \mathrm{E}\,\mathbf{x} \ \vdash \ B_0$$

then $\Gamma$ proves

$$\forall \mathbf{x}(A \wedge A_1 \to B_1) \wedge \ldots \wedge \forall \mathbf{x}(A \wedge A_n \to B_n) \ \vdash \ \forall \mathbf{x}(A \wedge A_0 \to B_0)$$

**Proposition 3.3.** *Sequent theories are well-formed.*

*Proof.* This is [Ru98, Corollary 4.14], a proof by induction on the complexity of derivations, where essentially the rules need a 'formalization'. One illustration: Suppose no variable of $\mathbf{y}$ is free in $A$. Since $A \wedge B \wedge \mathrm{E}\,\mathbf{y} \ \vdash \ A \wedge B$, we have $A \ \vdash \ \forall \mathbf{y}(B \to (A \wedge B))$. So $\forall \mathbf{xy}((A \wedge B) \to C) \wedge A \wedge \mathrm{E}\,\mathbf{x} \ \vdash \ \forall \mathbf{y}((A \wedge B) \to C) \wedge A \wedge \mathrm{E}\,\mathbf{x} \ \vdash \ \forall \mathbf{y}(B \to C)$. Thus Basic Predicate Logic proves

$$\forall \mathbf{xy}((A \wedge B) \to C) \ \vdash \ \forall \mathbf{x}(A \to \forall \mathbf{y}(B \to C)) \qquad \text{no variable of } \mathbf{y} \text{ free in } A$$

Other rules of BQC-E can be similarly formalized. □

Letter $p$ occurs *only formally* in $A[p]$ if all occurrences of $p$ are inside universal implication subformulas. In that case $A[p]$ is called a *formal context* of $p$. Context $A[p]$ is called a *proposition logical context* if $A[p]$ is built from $p$ and formulas without $p$ using only $\wedge$, $\vee$, and $\to$.

**Proposition 3.4.** *Let* $D[p]$ *be a formal proposition logical context of* $p$*. Then*

$$(A \leftrightarrow B) \wedge D[A] \ \vdash \ D[B]$$

*If* $C[p]$ *is a proposition logical context of* $p$*, then* $A \leftrightarrow B \ \vdash \ C[A] \leftrightarrow C[B]$*.*

*Proof.* See [Vi81, page 159] and [ArRu98, Proposition 2.5]. The particular case follows with $D[p]$ equal to $C[A] \leftrightarrow C[p]$. □

Proposition 3.4 has further extensions to predicate logic. For example:

**Proposition 3.5.** *Let* $A\mathbf{xz}$ *and* $B\mathbf{xz}$ *be formulas with all free variables among* $\mathbf{xz}$*, and* $D[p]$ *be a context with all free variables among* $\mathbf{xy}$ *and with* $p$ *not in the range of quantified variables from among* $\mathbf{xz}$*. Lists* $\mathbf{x}$*,* $\mathbf{y}$*, and* $\mathbf{z}$ *are pairwise disjoint. Then*

$$\forall \mathbf{xy}(A\mathbf{xz} \leftrightarrow B\mathbf{xz}) \ \vdash\ \forall \mathbf{xy}(D[A\mathbf{xz}] \leftrightarrow D[B\mathbf{xz}])$$

*Proof.* Suppose $\mathbf{c}$ is a list of new constant symbols and $A\mathbf{xc}\wedge \mathrm{E}\,\mathbf{x} \ \equiv\ B\mathbf{xc}\wedge \mathrm{E}\,\mathbf{x}$. We may suppose that no variable in $\mathbf{z}$ occurs in $A\mathbf{xc}$ or $B\mathbf{xc}$ or $D[p]$, if necessary after renaming of bound variables. By Proposition 3.1 we have $D[A\mathbf{xc}\wedge \mathrm{E}\,\mathbf{x}] \ \equiv\ D[B\mathbf{xc}\wedge \mathrm{E}\,\mathbf{x}]$. So with Proposition 3.3 we have $\forall \mathbf{xy}(A\mathbf{xc} \leftrightarrow B\mathbf{xc}) \ \vdash\ \forall \mathbf{xy}(D[A\mathbf{xc}\wedge \mathrm{E}\,\mathbf{x}] \leftrightarrow D[B\mathbf{xc}\wedge \mathrm{E}\,\mathbf{x}]) \ \equiv\ \forall \mathbf{xy}((D[A\mathbf{xc}\wedge \mathrm{E}\,\mathbf{x}]\wedge \mathrm{E}\,\mathbf{x}) \leftrightarrow (D[B\mathbf{xc}\wedge \mathrm{E}\,\mathbf{x}]\wedge \mathrm{E}\,\mathbf{x})) \ \equiv\ \forall \mathbf{xy}(D[A\mathbf{xc}] \leftrightarrow D[B\mathbf{xc}])$. We may suppose that the variables in $\mathbf{z}$ do not occur in the proofs. Replacing $\mathbf{c}$ by $\mathbf{z}$ gives a proof for $\forall \mathbf{xy}(A\mathbf{xz} \leftrightarrow B\mathbf{xz}) \ \vdash\ \forall \mathbf{xy}(D[A\mathbf{xz}] \leftrightarrow D[B\mathbf{xz}])$. □

Next a few elementary properties of quantifiers. Boldface lists $\mathbf{x}$ may have length 0, so may be empty. In such cases $\forall\mathbf{x}$ and $\exists\mathbf{x}$ are vacuous, and $\mathrm{E}\,\mathbf{x}$ is $\top$. Explicit variable cases like $\forall x$ or $\exists x$ or $\mathrm{E}\,x$ are not vacuous.

**Proposition 3.6.** *We have*

*K1.*

$\forall \mathbf{x}(B \to C) \ \vdash\ \forall \mathbf{x}(A \to B^C)$ *and*

$\forall \mathbf{x}(A \to B) \ \vdash\ \forall \mathbf{x}((A\wedge C) \to (B\wedge C))$ *and*

$\forall \mathbf{x}(A \to B) \ \vdash\ \forall \mathbf{x}(B^C \to A^C)$

*K2.* $\forall \mathbf{xy}(A \to B) \ \equiv\ \forall \mathbf{x}(\exists \mathbf{y} A \to B)$ *no variable in* $\mathbf{y}$ *free in* $B$

*Proof.* Case K1. First entailment: $\forall \mathbf{x}(B \to C)\wedge A\wedge \mathrm{E}\,\mathbf{x} \ \vdash\ ((B\wedge \mathrm{E}\,\mathbf{x}) \to C)\wedge A\wedge \mathrm{E}\,\mathbf{x} \ \vdash\ B \to C$. Second entailment: $\forall \mathbf{x}(A \to B) \ \vdash\ \forall \mathbf{x}((A\wedge C) \to B)\wedge \forall \mathbf{x}((A\wedge C) \to C)$. Third entailment: $\forall \mathbf{x}(A \to B)\wedge B^C\wedge \mathrm{E}\,\mathbf{x} \ \vdash\ ((A\wedge \mathrm{E}\,\mathbf{x}) \to B)\wedge B^C\wedge \mathrm{E}\,\mathbf{x} \ \vdash\ A^C$.

Case K2. The nontrivial direction, $\forall \mathbf{x}(\exists \mathbf{y} A \to B) \ \vdash\ \forall \mathbf{xy}(\exists \mathbf{y} A \to B)\wedge \forall \mathbf{xy}(A \to \exists \mathbf{y} A) \ \vdash\ \forall \mathbf{xy}(A \to B)$. □

Our predicate logic allows for structures without element, still an uncommon practice since in classical predicate logic CQC there is no need for empty structures. So we use extra caution with statements that may require existence. Sentence $\exists x\top$ corresponds with structures that have an element. The following proposition is an illustration.

**Proposition 3.7.** *We have*

*K3.* $\exists xy\top \ \equiv\ \exists z\top$ *and* $\forall \mathbf{x}\,\top^{aa} \ \equiv\ (\exists \mathbf{x}\top^a)^a$

*K4.* $\exists \mathbf{x}\, a \ \vdash\ a$ *and* $a\wedge \exists \mathbf{x}\top \ \equiv\ \exists \mathbf{x}\, a$

*K5.* $\dfrac{A\wedge \exists y\top \ \vdash\ B}{\exists \mathbf{x}A \ \vdash\ B}$ $\mathbf{x}$ *has positive length, and none free in* $B$

*K6.* $\exists x\top \wedge \forall \mathbf{y}(B \to C) \ \vdash\ \exists \mathbf{y}(B \to C)$

*Proof.* Case K3. $\mathrm{E}\,x\wedge \mathrm{E}\,y\wedge \top \ \vdash\ \mathrm{E}\,x\wedge \exists y\top \ \vdash\ \exists xy\top$. Substitution of $z$ for $x$ and $y$ gives $\mathrm{E}\,z\wedge \top \ \vdash\ \exists xy\top$. Thus $\exists z\top \ \vdash\ \exists xy\top$. And so on, any nonempty sequence of variables works. The second equivalence follows from K2.

Case K4. Easy.

Case K5. We may suppose $y$ is a new variable. We have $A\wedge \mathrm{E}\,\mathbf{x}y \ \vdash\ A\wedge \exists y\top$. Substitute one of the variables of $\mathbf{x}$ for $y$.

Case K6. $\mathrm{E}\,\mathbf{y}\wedge \forall \mathbf{y}(B \to C) \ \vdash\ \mathrm{E}\,\mathbf{y}\wedge (B \to C) \ \vdash\ \exists \mathbf{y}(B \to C)$. So $\exists x\top\wedge \forall \mathbf{y}(B \to C) \ \vdash\ \exists \mathbf{y}\top\wedge \forall \mathbf{y}(B \to C) \ \equiv\ \exists \mathbf{y}(\forall \mathbf{y}(B \to C)) \ \vdash\ \exists \mathbf{y}(B \to C)$. □

**Proposition 3.8.** *We have*

*K7.* $\forall \mathbf{x}(\top \to A)\wedge \forall \mathbf{x}\, A^{aa} \ \equiv\ \forall \mathbf{x}(\top \to A)\wedge \forall \mathbf{x}\,\top^{aa}$ *equivalently*

$\forall \mathbf{x}(\top \to A)\wedge (\exists \mathbf{x} A^a)^a \ \equiv\ \forall \mathbf{x}(\top \to A)\wedge (\exists \mathbf{x}\top^a)^a$

*K8.* $\forall\mathbf{x}\, A^{aaa} \;\equiv\; \forall\mathbf{x}\, A^{a} \wedge \forall\mathbf{x}\, \top^{aa}$ *equivalently*

$(\exists\mathbf{x} A^{aa})^{a} \;\equiv\; (\exists\mathbf{x} A)^{a} \wedge (\exists\mathbf{x} \top^{a})^{a} \;\equiv\; (\exists\mathbf{x}(A \vee \top^{a}))^{a}$

*K9.* $\forall\mathbf{x}\, A^{aaaa} \;\equiv\; \forall\mathbf{x}\, A^{aa}$ *equivalently*

$(\exists\mathbf{x} A^{aaa})^{a} \;\equiv\; (\exists\mathbf{x} A^{a})^{a}$

*Proof.* Case K7. From left to right follows with $\vdash\ \forall\mathbf{x}(\top^{a} \to A^{a})$. From right to left, by K1 we have $\forall\mathbf{x}(\top \to A) \;\vdash\; \forall\mathbf{x}(A^{a} \to \top^{a})$.

Case K8. From right to left, with K1 we get $\forall\mathbf{x}\, A^{a} \wedge \forall\mathbf{x}\, \top^{aa} \;\vdash\; \forall\mathbf{x}(A^{aa} \to \top^{a}) \wedge \forall\mathbf{x}\, \top^{aa} \;\vdash\; \forall\mathbf{x}\, A^{aaa}$. Left to right: First, with $\vdash\ \forall\mathbf{x}(\top^{a} \to A^{aa})$ we get $\forall\mathbf{x}\, A^{aaa} \;\vdash\; \forall\mathbf{x}\, \top^{aa} \;\vdash\; \forall\mathbf{x}(A \to (A \wedge \top^{aa}))$. Second, with $\vdash\ \forall\mathbf{x}((A \wedge \top^{aa}) \to A^{aa})$ we get $\forall\mathbf{x}\, A^{aaa} \;\vdash\; \forall\mathbf{x}\, (A \wedge \top^{aa})^{a}$. Thus $\forall\mathbf{x}\, A^{aaa} \;\vdash\; \forall\mathbf{x}\, A^{a}$.

Case K9. Substitute $A \mapsto A^{a}$ in K8. □

The following are proposition logical immediate consequences of Proposition 3.8.

$(\top \to A) \wedge A^{aa} \;\equiv\; (\top \to A) \wedge \top^{aa}$ so also
$A \;\vdash\; \top^{aa}$ exactly when $A \;\vdash\; A^{aa}$
$A^{aaa} \;\equiv\; A^{a} \wedge \top^{aa} \;\equiv\; (A \vee \top^{a})^{a}$
$A^{aaaa} \;\equiv\; A^{aa}$

**Proposition 3.9.** *We have*

*K10.* $\forall\mathbf{x}\, (A \wedge B)^{a} \;\vdash\; \forall\mathbf{x}(A \to B^{a}) \;\vdash\; \forall\mathbf{x}((A \wedge B) \to \top^{a})$

*K11.* $\forall\mathbf{x}\, (A \wedge B)^{aa} \;\equiv\; \forall\mathbf{x}\, A^{aa} \wedge \forall\mathbf{x}\, B^{aa} \;\equiv\; \forall\mathbf{x}\, (A \to B^{a})^{a} \;\equiv\; \forall\mathbf{x}\, ((A \wedge B) \to \top^{a})^{a}$

*K12.* $A \;\vdash\; \forall\mathbf{x}\, A^{aa}$ *plus* $B \;\vdash\; \forall\mathbf{x}\, B^{aa}$ *implies* $A \vee B \;\vdash\; \forall\mathbf{x}\, (A \vee B)^{aa}$

*K13.* $\forall\mathbf{x}\, A^{aa} \wedge \forall\mathbf{x}\, (B \vee C)^{aa} \;\equiv\; \forall\mathbf{x}\, ((A \wedge B) \vee (A \wedge C))^{aa}$

*K14.* $A \wedge \forall\mathbf{x}\, A^{a} \;\vdash\; A \wedge \forall\mathbf{x}\, \top^{a}$ *so* $\forall\mathbf{x}\, \top^{a} \;\vdash\; A$ *implies* $A \wedge \forall\mathbf{x}\, A^{a} \;\equiv\; \forall\mathbf{x}\, \top^{a}$

*K15.* $\forall\mathbf{x}\, (A \vee A^{a})^{a} \;\equiv\; \forall\mathbf{x}\, \top^{a}$

*K16.* $\forall\mathbf{x}\, A^{aaa} \wedge \forall\mathbf{x}\, B^{aaa} \;\equiv\; \forall\mathbf{x}\, (A \vee B)^{aaa} \;\equiv\; \forall\mathbf{x}\, (A^{aa} \vee B^{aa})^{a} \;\equiv\; \forall\mathbf{x}\, (A \vee B \vee \top^{a})^{a}$
*so* $\forall\mathbf{x}\, (A^{aa} \vee B^{aa})^{aa} \;\equiv\; \forall\mathbf{x}\, (A \vee B)^{aa}$

*K17.* $\forall\mathbf{x}((A^{aa} \vee B^{aa}) \to C^{a}) \;\equiv\; \forall\mathbf{x}((A \vee B)^{aa} \to C^{a})$

*K18.* $\forall\mathbf{x}(A \to C) \vee \forall\mathbf{x}(A \to B) \;\vdash\; \forall\mathbf{x}(A \to (C \vee B))$
*so* $\forall\mathbf{x}\, (A \to (C \vee B))^{a} \;\vdash\; \forall\mathbf{x}\, ((A \to C) \vee (A \to B))^{a} \;\vdash\; \forall\mathbf{x}\, ((A \to C) \vee B)^{a}$

*K19.* $\forall\mathbf{x}\, (A \to (a \vee B))^{a} \;\equiv\; \forall\mathbf{x}\, (A^{a} \vee B)^{a} \;\equiv\; \forall\mathbf{x}\, (A^{a} \vee (a \vee B))^{a}$

*K20.* $\forall\mathbf{x}\, (A \to (a \vee B))^{a} \;\equiv\; \forall\mathbf{x}\, (A^{aa} \to B^{aa})^{a}$

*Proof.* Case K10. First entailment, use $\forall\mathbf{x}\, (A \wedge B)^{a} \wedge A \wedge \mathrm{E}\,\mathbf{x} \;\vdash\; (A \wedge B)^{a} \wedge A \wedge \mathrm{E}\,\mathbf{x} \;\vdash\; B^{a}$. Second entailment, use $\vdash\ \forall\mathbf{x}((A \wedge B) \to A)$ and $\vdash\ \forall\mathbf{x}(B^{a} \to (A \wedge B)^{a})$.

Case K11. In K10, by setting $\mathbf{x}$ to length 0 we get $(A \to B^{a}) \;\vdash\; ((A \wedge B) \to \top^{a})$, and so on. So the schemas of K10 imply $\vdash\ \forall\mathbf{x}((A \to B^{a}) \to ((A \wedge B) \to \top^{a}))$ and so on. The right to left entailments between first, third, and fourth formulas follow from K10. From first to second, use $\vdash\ \forall\mathbf{x}(A^{a} \to (A \wedge B)^{a})$ and $\vdash\ \forall\mathbf{x}(B^{a} \to (A \wedge B)^{a})$. From second to third: $\forall\mathbf{x}\, A^{aa} \;\vdash\; \forall\mathbf{x}\, \top^{aa}$ and $\forall\mathbf{x}\, A^{aa} \wedge \forall\mathbf{x}\, B^{aa} \wedge (A \to B^{a}) \wedge \mathrm{E}\,\mathbf{x} \;\vdash\; A^{aa} \wedge B^{aa} \wedge (A \to B^{a}) \wedge \mathrm{E}\,\mathbf{x} \;\vdash\; \top^{a}$. From first to fourth: $\forall\mathbf{x}\, (A \wedge B)^{aa} \;\vdash\; \forall\mathbf{x}\, \top^{aa} \;\vdash\; \forall\mathbf{x}(((A \wedge B) \to \top^{a}) \to (A \wedge B)^{a})$.

Case K12. We have $\vdash\ \forall\mathbf{x}((A \vee B)^{a} \to A^{a})$ and $\vdash\ \forall\mathbf{x}((A \vee B)^{a} \to B^{a})$.

Case K13. By K11 we have $\forall\mathbf{x}\, A^{aa} \wedge \forall\mathbf{x}\, (B \vee C)^{aa} \;\equiv\; \forall\mathbf{x}\, (A \wedge (B \vee C))^{aa}$.

Case K14. Immediate from Proposition 3.1.

Case K15. We have $\forall \mathbf{x}\, A^{aa} \wedge \forall \mathbf{x}\, A^{a} \;\vdash\; \forall \mathbf{x}\,(A^{a} \wedge \mathrm{E}\,\mathbf{x})^{a} \wedge (A \wedge \mathrm{E}\,\mathbf{x})^{a} \;\vdash\; \forall \mathbf{x}\,(\mathrm{E}\,\mathbf{x})^{a} \;\equiv\; \forall \mathbf{x}\, \top^{a}$.

Case K16. $\forall \mathbf{x}\, A^{aaa} \wedge \forall \mathbf{x}\, B^{aaa} \;\equiv\; \forall \mathbf{x}\,(A^{a} \wedge B^{a})^{aa} \;\equiv\; \forall \mathbf{x}\,(A^{a} \wedge B^{a})^{aaaa}$. See K8 and K9 and K11.

Case K17. $A^{aa} \vee B^{aa} \;\vdash\; (A \vee B)^{aa}$ implies the right to left direction. Conversely, $\forall \mathbf{x}((A^{aa} \vee B^{aa}) \to C^{a}) \;\vdash\; \forall \mathbf{x}((A^{aa} \vee B^{aa})^{aa} \to C^{aaa}) \;\vdash\; \forall \mathbf{x}((A \vee B)^{aa} \to C^{a})$.

Case K18. Immediate from Proposition 3.2.

Case K19. First equivalence, difficult direction: $\forall \mathbf{x}\,(A^{a} \vee B)^{a} \;\equiv\; \forall \mathbf{x}\, A^{aa} \wedge \forall \mathbf{x}\, B^{a} \;\vdash\; \forall \mathbf{x}\,((A \to (a \vee B)) \wedge (a \vee B)^{a})^{a} \wedge \forall \mathbf{x}\, B^{a} \;\equiv\; \forall \mathbf{x}\,((A \to (a \vee B)) \wedge a^{a} \wedge B^{a})^{a} \wedge \forall \mathbf{x}\, B^{a} \;\equiv\; \forall \mathbf{x}\,(A \to (a \vee B))^{a} \wedge \forall \mathbf{x}\, B^{a}$. The second equivalence follows from the first with substitution $B \mapsto (a \vee B)$.

Case K20. With K19 and K16 we have $\forall \mathbf{x}\,(A \to (a \vee B))^{a} \;\equiv\; \forall \mathbf{x}\,(A^{a} \vee B)^{a} \;\equiv\; \forall \mathbf{x}\,(A^{a} \vee B \vee \top^{a})^{a} \;\equiv\; \forall \mathbf{x}\,(A^{aaa} \vee B^{aa})^{a} \;\equiv\; \forall \mathbf{x}\,(A^{aa} \to (a \vee B^{aa}))^{a} \;\equiv\; \forall \mathbf{x}\,(A^{aa} \to B^{aa})^{a}$. □

Here are some proposition logical consequences of Proposition 3.9.

$(A \wedge B)^{a} \;\vdash\; A \to B^{a} \;\vdash\; (A \wedge B) \to \top^{a}$
$(A \wedge B)^{aa} \;\equiv\; A^{aa} \wedge B^{aa} \;\equiv\; (A \to B^{a})^{a} \;\equiv\; ((A \wedge B) \to \top^{a})^{a}$
$A \;\vdash\; A^{aa}$ plus $B \;\vdash\; B^{aa}$ implies $A \vee B \;\vdash\; (A \vee B)^{aa}$
$A^{aa} \wedge (B \vee C)^{aa} \;\equiv\; ((A \wedge B) \vee (A \wedge C))^{aa}$
$A \wedge A^{a} \;\vdash\; A \wedge \top^{a}$ so also $\top^{a} \;\vdash\; A$ implies $A \wedge A^{a} \;\equiv\; \top^{a}$
$(A \vee A^{a})^{a} \;\equiv\; \top^{a}$ so also $(A \vee A^{a})^{aa} \;\equiv\; \top^{aa}$
$A^{aaa} \wedge B^{aaa} \;\equiv\; (A \vee B)^{aaa} \;\equiv\; (A^{aa} \vee B^{aa})^{a} \;\equiv\; (A \vee B \vee \top^{a})^{a}$ so
$\quad (A^{aa} \vee B^{aa})^{aa} \;\equiv\; (A \vee B)^{aa}$
$(A^{aa} \vee B^{aa}) \to C^{a} \;\equiv\; (A \vee B)^{aa} \to C^{a}$
$(A \to C) \vee (A \to B) \;\vdash\; (A \to (C \vee B))$ so
$\quad (A \to (C \vee B))^{a} \;\vdash\; ((A \to C) \vee (A \to B))^{a} \;\vdash\; ((A \to C) \vee B)^{a}$
$(A \to (a \vee B))^{a} \;\equiv\; (A^{a} \vee B)^{a} \;\equiv\; (A^{a} \vee (a \vee B))^{a} \;\equiv\; (A^{aa} \to B^{aa})^{a}$

**Proposition 3.10.** *We have*

*K21.* $A \wedge \mathrm{E}\,x \;\vdash\; (A \wedge \mathrm{E}\,x)^{aa}$ *implies* $\exists x A \;\vdash\; (\exists x A)^{aa}$

*K22.* $x$ *not free in* $A$ *implies* $A^{aa} \wedge (\exists x B)^{aa} \;\equiv\; (\exists x(A \wedge B))^{aa}$

*K23.* $(\exists \mathbf{x} A^{aa})^{a} \;\equiv\; (\exists \mathbf{x}(A \vee \top^{a}))^{a}$ *and* $(\exists \mathbf{x} A^{aa})^{aa} \;\equiv\; (\exists \mathbf{x} A)^{aa}$

*K24.* $\exists \mathbf{x} A^{aa} \;\vdash\; (\exists \mathbf{x} A)^{aa}$ *and* $\exists \mathbf{x} A^{aa} \to C^{a} \;\equiv\; (\exists \mathbf{x} A)^{aa} \to C^{a}$

*Proof.* Case K21. $(A \wedge \mathrm{E}\,x)^{aa} \;\vdash\; (\exists x A)^{aa}$.

Case K22. $A^{aa} \wedge (\exists x B)^{aa} \;\equiv\; (A \wedge \exists x B)^{aa} \;\equiv\; (\exists x(A \wedge B))^{aa}$.

Case K23. The first equivalence is K8. So $(\exists \mathbf{x} A^{aa})^{aa} \;\equiv\; (\exists \mathbf{x} A \vee \exists \mathbf{x} \top^{a})^{aa}$. Now $(\exists \mathbf{x} A)^{a} \;\vdash\; \top$ gives $\exists \mathbf{x} \top^{a} \;\vdash\; (\exists \mathbf{x} A)^{aa}$. Thus with K16 we have $(\exists \mathbf{x} A \vee \exists \mathbf{x} \top^{a})^{aa} \;\equiv\; ((\exists \mathbf{x} A)^{aa} \vee (\exists \mathbf{x} \top^{a})^{aa})^{aa} \;\equiv\; (\exists \mathbf{x} A)^{aa}$.

Case K24. First, $A^{aa} \wedge \mathrm{E}\,\mathbf{x} \;\vdash\; A^{aa} \wedge \top^{aa} \wedge \mathrm{E}\,\mathbf{x} \;\vdash\; A^{aa} \wedge (\mathrm{E}\,\mathbf{x})^{aa} \;\vdash\; (\exists \mathbf{x} A)^{aa}$. Second, $\exists \mathbf{x} A^{aa} \to C^{a} \;\vdash\; (\exists \mathbf{x} A^{aa})^{aa} \to C^{aaa} \;\vdash\; (\exists \mathbf{x} A^{aa})^{aa} \to C^{a} \;\equiv\; (\exists \mathbf{x} A)^{aa} \to C^{a}$. □

**Proposition 3.11.** *We have*

*K25.* $\forall \mathbf{x}(A^{aa} \to B^{aa}) \;\equiv\; \forall \mathbf{x}((B^{a} \wedge \top^{aa}) \to A^{a})$

*K26.* $\forall \mathbf{x}\,(A \to B)^{aa} \wedge \forall \mathbf{x}\, B^{a} \;\vdash\; \forall \mathbf{x}\, A^{a}$

*K27.* $\forall \mathbf{x}\,(A \to B)^{aa} \;\vdash\; \forall \mathbf{x}\,(A^{aa} \to B^{aa})^{aa}$
*but* $(A^{aa} \to B^{aa})^{aa} \;\nvdash\; (A \to B)^{aa}$*, not even over* IQC

*Proof.* Case K25. $\forall\mathbf{x}(A^{aa} \to B^{aa}) \ \vdash \ \forall\mathbf{x}(B^{aaa} \to A^{aaa}) \ \equiv \ \forall\mathbf{x}((B^a \wedge \top^{aa}) \to A^a) \ \vdash \ \forall\mathbf{x}(A^{aa} \to (B^a \wedge \top^{aa})^a) \ \equiv \ \forall\mathbf{x}(A^{aa} \to B^{aa})$.

Case K26. $\forall\mathbf{x}\,(A \to B)^{aa} \wedge \forall\mathbf{x}\,B^a \ \vdash \ \forall\mathbf{x}\,(A \to (a \vee B))^{aa} \wedge \forall\mathbf{x}\,B^a \ \equiv \ \forall\mathbf{x}\,(A^a \vee B)^{aa} \wedge \forall\mathbf{x}\,B^a \ \equiv \ \forall\mathbf{x}\,(A^{aa} \wedge B^a)^a \wedge \forall\mathbf{x}\,B^a$. We have

$$\begin{array}{l} \forall\mathbf{x}\,(A^{aa} \wedge B^a)^a \wedge \forall\mathbf{x}\,B^a \ \vdash \\ \forall\mathbf{x}\,(A^{aa} \wedge B^a)^a \wedge \forall\mathbf{x}(\top \to B^a) \ \vdash \\ \forall\mathbf{x}\,(A^{aa} \wedge B^a)^a \wedge \forall\mathbf{x}(A^{aa} \to (A^{aa} \wedge B^a)) \end{array}$$

So we can continue with equivalence $\forall\mathbf{x}\,(A^{aa} \wedge B^a)^a \wedge \forall\mathbf{x}\,B^a \ \equiv \ \forall\mathbf{x}\,A^{aaa} \wedge \forall\mathbf{x}\,B^a \ \vdash \ \forall\mathbf{x}\,A^a$. Another proof: $\forall\mathbf{x}(A \to B) \wedge \forall\mathbf{x}\,B^a \ \vdash \ \forall\mathbf{x}\,A^a$, so $\forall\mathbf{x}\,(A \to B)^{aa} \wedge \forall\mathbf{x}\,B^{aaa} \ \vdash \ \forall\mathbf{x}\,A^{aaa}$. Thus $\forall\mathbf{x}\,(A \to B)^{aa} \wedge \forall\mathbf{x}\,B^a \ \vdash \ \forall\mathbf{x}\,(A \to B)^{aa} \wedge \forall\mathbf{x}\,\top^{aa} \wedge \forall\mathbf{x}\,B^a \ \vdash \ \forall\mathbf{x}\,A^{aaa} \ \vdash \ \forall\mathbf{x}\,A^a$.

Case K27. The entailment follows from K26 and K9 and K1 with $(A \to B)^{aa} \wedge A^{aa} \ \vdash \ B^{aa}$ and $(A^{aa} \to B^{aa})^a \ \vdash \ (A \to B)^{aaa}$ and $\forall\mathbf{x}\,(A \to B)^{aa} \ \equiv \ \forall\mathbf{x}\,(A \to B)^{aaaa}$. For the counterexample set $A = a$ and $B = \bot$. So show $(\top^a \to \top^a)^{aa} \ \nvdash \ (a \to \bot)^{aa}$, which over IQC is equivalent to $\nvdash \ (\neg a)^{aa} \ \equiv \ (\neg\neg a) \to a$. This follows from a two-node reflexive transitive Kripke model. □

Some proposition logical consequences of Proposition 3.11.

$$\begin{array}{l} A^{aa} \to B^{aa} \ \equiv \ (B^a \wedge \top^{aa}) \to A^a \\ (A \to B)^{aa} \wedge B^a \ \vdash \ A^a \\ (A \to B)^{aa} \ \vdash \ (A^{aa} \to B^{aa})^{aa} \end{array}$$

**Proposition 3.12.** *We have*

*K28.* $(\forall\mathbf{x}(A \to B))^{aa} \ \vdash \ \forall\mathbf{x}\,(A \to B)^{aa}$

*K29.* $(\forall\mathbf{x}\,(A \to B)^{aa})^{aa} \ \nvdash \ (\forall\mathbf{x}(A \to B))^{aa}$, *not even over* IQC

*K30.* $\forall\mathbf{x}\,(A \to B^{aa})^{aa} \ \vdash \ \forall\mathbf{x}(A \to B^{aa})$

*K31.* $(\forall\mathbf{x}\,(A \to B^{aa})^{aa})^a \ \equiv \ (\forall\mathbf{x}(A \to B^{aa}))^a$

*Proof.* Case K28. We have $(\forall\mathbf{x}(A \to B))^{aa} \ \vdash \ \top^{aa}$ and $(\forall\mathbf{x}(A \to B))^{aa} \wedge (A \to B)^a \wedge \mathrm{E}\,\mathbf{x} \ \vdash \ (A \to B)^{aa} \wedge (A \to B)^a \wedge \mathrm{E}\,\mathbf{x} \ \vdash \ \top^a$. So $(\forall\mathbf{x}(A \to B))^{aa} \ \vdash \ \forall\mathbf{x}((A \to B)^a \to \top^a) \ \equiv \ \exists\mathbf{x}(A \to B)^a \to \top^a$. Thus $(\forall\mathbf{x}(A \to B))^{aa} \ \vdash \ \forall\mathbf{x}\,(A \to B)^{aa}$.

Case K29. Consider the reflexive transitive Kripke model $\mathfrak{A}$ for IQC with as underlying poset a linearly ascending list of nodes $\alpha_1, \alpha_2, \alpha_3, \ldots$ with domains $D(\alpha_n) = \{1, 2, 3, 4, , \ldots, n\}$ and $\alpha_n \Vdash Q(m)$ exactly when $n > m$. Set $A = \top$ and $B = Q(x)$ and $a = \bot$. Then $\mathfrak{A} \Vdash \forall x \neg\neg Q(x)$ and $\mathfrak{A} \Vdash \neg\forall x Q(x)$.

Case K30. $\forall\mathbf{x}\,(A \to B^{aa})^{aa} \wedge A \wedge \mathrm{E}\,\mathbf{x} \vdash (A \to B^{aa})^{aa} \wedge A \wedge \mathrm{E}\,\mathbf{x} \vdash (\top \to B^{aa})^{aa} \ \equiv \ (\top \to (a \vee B^{aa}))^{aa} \ \equiv \ (\top^a \vee B^{aa})^{aa} \ \equiv \ B^{aaaa} \ \equiv \ B^{aa}$.

Case K31. Write $X \ = \ \forall\mathbf{x}\,(A \to B^{aa})^{aa}$ and $Y \ = \ \forall\mathbf{x}(A \to B^{aa})$. With K28 and K30 we have $Y^{aa} \ \vdash \ X \ \vdash \ Y$. So $X^a \ \vdash \ Y^{aaa} \ \vdash \ Y^a \ \vdash \ X^a$. □

A two-node irreflexive Kripke model shows that $\top \ \nvdash \ \top^{aa}$. So case K30 is strictly one way (set $A$ equal to $B^{aa}$). Some proposition logical consequences of Proposition 3.12.

$$\begin{array}{ll} (A \to B^{aa})^{aa} \ \vdash \ (A \to B^{aa}) & \text{even } (A \to B^{aa})^{aa} \wedge A \ \equiv \ B^{aa} \wedge A \\ (A \to B^{aa})^{aaa} \ \equiv \ (A \to B^{aa})^a & \end{array}$$

# 4 Kolmogorov and Kuroda Translations

The Kolmogorov translation of [Ko25] is for an embedding of Classical Predicate Logic CQC into Intuitionistic Predicate Logic IQC. Our translation includes an embedding into Basic Predicate Logic BQC-E. Kolmogorov replaced each subformula $A$ by $\neg\neg A = (A \to \bot) \to \bot$. We broaden this by replacing the subformulas by $(A \to a) \to a$, also written as $A^{aa}$. There is some connection with the Friedman Dragalin translation $A \mapsto A \cdot a$, definable by replacing all atoms $P$, which includes $\top$ and $\bot$, by $(a \vee P)$.

Our *Kolmogorov* translation $A \mapsto k_a(A)$, or simply $k(A)$, is inductively defined by

$$
\begin{aligned}
k(P) &:= P^{aa} \quad \text{for atoms } P \\
k(A \wedge B) &:= (k(A) \wedge k(B))^{aa} \\
k(A \vee B) &:= (k(A) \vee k(B))^{aa} \\
k(\exists x A) &:= (\exists x\, k(A))^{aa} \\
k(\forall \mathbf{x}(A \to B)) &:= (\forall \mathbf{x}\,(k(A) \to k(B))^{aa})^{aa}
\end{aligned}
$$

Since $\top$, $\bot$, and $a$ are atoms, we have $k(\top) = \top^{aa}$ and $k(\bot) \equiv k(a) \equiv \top^{a}$. Similarly, $k(\mathrm{E}\,x) \equiv (\mathrm{E}\,x)^{aa}$. Obviously $\top^{a} \vdash k(A) \vdash \top^{aa}$ and $k(A)^{aa} \equiv k(A)$, and $k(A \cdot a) \equiv k(A) \cdot a \equiv k(A)$ since $(a^{aa} \vee P^{aa})^{aa} \equiv (a \vee P)^{aa} \equiv P^{aa}$. Positive occurrences of subformulas are mapped to positive occurrences, and negative occurrences of subformulas are mapped to negative occurrences. Formulas $A$ and $k(A)$ have the same free variables, and $k(A)[y/t]$ equals $k(A[y/t])$. With K23 we have $k(\exists x \exists y A) = (\exists x\, k(\exists y A))^{aa} = (\exists x (\exists y\, k(A))^{aa})^{aa} \equiv (\exists xy\, k(A))^{aa}$ and so on. So $k(\exists \mathbf{x} A) \equiv (\exists \mathbf{x}\, k(A))^{aa}$ for all $\mathbf{x}$.

We defined $k(A \to B)$ essentially equal to $(k(A) \to k(B))^{aaaa}$, which by K9 is equivalent to $(k(A) \to k(B))^{aa}$. More generally, for universal implication there is some ambiguity in what it means to replace each subformula $A$ by $A^{aa}$. Is $\forall \mathbf{x}(A \to B)$ a 'double' formula where the $aa$ substitution should be applied twice as done above, or is it a single construction where the $aa$ substitution should be applied once? The following Proposition shows that the two versions are equivalent.

**Proposition 4.1.** $k(\forall \mathbf{x}(A \to B)) \equiv (\forall \mathbf{x}(k(A) \to k(B)))^{aa}$.

*Proof.* By $k(B) \equiv k(B)^{aa}$ and K31. □

The following function is based on the one in [Ku51]. The *Kuroda* translation $A \mapsto r_a(A)$, or simply $r(A)$, is defined by $r(A) := u_a(A)^{aa}$ with $u = u_a$ inductively defined by

$$
\begin{aligned}
u(P) &:= (a \vee P) \quad \text{for atoms } P \\
u(A \wedge B) &:= (u(A) \wedge u(B)) \\
u(A \vee B) &:= (u(A) \vee u(B)) \\
u(\exists x A) &:= (a \vee \exists x\, u(A)) \\
u(\forall \mathbf{x}(A \to B)) &:= (\forall \mathbf{x}\,(u(A) \to u(B))^{aa})
\end{aligned}
$$

If $a = \bot$ as in Kuroda's original, then the $a\vee$ can be dropped in the definitions of $u(P)$ and $u(\exists x A)$. Following the proof of Proposition 4.3, we consider the equivalent variant where we define $u(A \to B) := (u(A) \to u(B))$ when $\mathbf{x}$ is empty in the $\forall$ definition step. We show below that the Kuroda translation is essentially the same as the Kolmogorov translation.

**Proposition 4.2.** *$a \vdash u(A)$ for all $A$.*

*Proof.* A trivial proof by induction on the complexity of $A$. For example $a \vdash \forall \mathbf{x}\, A^{a}$ covers universal implication. □

Rule $u(\exists xA) \;:=\; (a \vee \exists x\, u(A))$ needs the extra $a\vee$, for otherwise Proposition 4.2 fails for $\exists x\, u(P) \;\equiv\; \exists x(a \vee P)$. In general we only have $\exists x\, a \;\equiv\; a \wedge \exists x\top \;\vdash\; \exists x\, u(A)$. Example derivation: From $a \;\vdash\; u(A)$ we only get $a \wedge \mathrm{E}\, x \;\vdash\; u(A) \wedge \mathrm{E}\, x \;\vdash \exists x\, u(A)$. As we mentioned immediately before Section 3, if we remove $\ell$ and its axioms from BQC-E and add $\vdash \mathrm{E}\, x$, we essentially get BQC of [Ru98]. In that case we don't need $a\vee$ and can define $u(\exists xA) \;:=\; \exists x\, u(A)$.

We have $r(A)^{aa} \equiv r(A)$, and $r(A \cdot a) \equiv r(A) \cdot a \equiv r(A)$ and $u(\exists \mathbf{x}A) \equiv (a \vee \exists \mathbf{x}\, u(A))$. Positive occurrences of subformulas are mapped to positive occurrences, and negative occurrences of subformulas are mapped to negative occurrences. Formulas $A$ and $r(A)$ have the same free variables, and $r(A)[y/t]$ equals $r(A[y/t])$.

**Proposition 4.3.** *We have*

*K32.* $k(P) \equiv r(P)$ *for all atoms* $P$ *even* $P^a \;\equiv\; u(P)^a$

*K33.* $k(A) \equiv r(A')$ *plus* $k(B) \equiv r(B')$ *implies* $k(A \wedge B) \;\equiv\; r(A' \wedge B')$

*K34.* $k(A) \equiv r(A')$ *plus* $k(B) \equiv r(B')$ *implies* $k(A \vee B) \;\equiv\; r(A' \vee B')$

*K35.* $k(A) \equiv r(A')$ *implies* $k(\exists xA) \;\equiv\; r(\exists xA')$

*K36.* $k(A) \equiv r(A')$ *plus* $k(B) \equiv r(B')$ *implies* $k(\forall \mathbf{x}(A \to B)) \;\equiv\; r(\forall \mathbf{x}(A' \to B'))$

*K37.* $k(A) \equiv r(A)$ *for all* $A$

*Proof.* Case K32. $P^a \;\equiv\; (a \vee P)^a$.

Case K33. $k(A \wedge B) \;\equiv\; k(A) \wedge k(B) \;\equiv\; r(A') \wedge r(B') \;=\; u(A')^{aa} \wedge u(B')^{aa} \;\equiv\; (u(A') \wedge u(B'))^{aa} \;=\; r(A' \wedge B')$.

Case K34. With K16 we have $k(A \vee B) \;=\; (k(A) \vee k(B))^{aa} \;\equiv\; (r(A') \vee r(B'))^{aa} \;=\; (u(A')^{aa} \vee u(B')^{aa})^{aa} \;\equiv\; (u(A') \vee u(B'))^{aa} \;=\; r(A' \vee B')$.

Case K35. With K23 and schema $(a \vee X)^a \;\equiv\; X^a$ we have $k(\exists xA) \;=\; (\exists x\, k(A))^{aa} \;\equiv\; (\exists x\, r(A'))^{aa} \;\equiv\; (\exists x\, u(A'))^{aa} \;\equiv\; (a \vee \exists x\, u(A'))^{aa} \;=\; (u(\exists xA'))^{aa} \;=\; r(\exists xA')$.

Case K36. With Proposition 4.2 and K20 we have $k(\forall \mathbf{x}(A \to B)) \;\equiv\; (\forall \mathbf{x}\, (u(A')^{aa} \to u(B')^{aa})^{aa})^{aa} \;\equiv\; (\forall \mathbf{x}\, (u(A') \to u(B'))^{aa})^{aa} \;=\; r(\forall \mathbf{x}(A' \to B'))$.

Case K37. By induction on the complexity of $A$. □

We stay slightly closer to Kuroda's original version if we separately define $u(A \to B) \;:=\; (u(A) \to u(B))$ when $\mathbf{x}$ is empty in the $\forall$ definition step. With this modification we get $k(A \to B) \;\equiv\; (k(A) \to k(B))^{aaaa} \;\equiv\; (k(A) \to k(B))^{aa} \;\equiv\; (u(A')^{aa} \to u(B')^{aa})^{aa} \;\equiv\; (u(A') \to u(B'))^{aa} \;=\; r(A' \to B')$ as alternate special case of K36. With this inessential modification, Kuroda's translation with $a = \bot$ is a predicate logic extension of Glivenko's proposition logical translation of [Gl29]. So with $a = \bot$, and for all formulas $A$ without any occurrence of $\forall \mathbf{x}$ with non-empty $\mathbf{x}$, we have $r(A) \;\equiv\; \neg\neg A$.

# 5 K Embeddings

The Kolmogorov and Kuroda translations are essentially equivalent. So we need only one version of an embedding of Classical Predicate Logic into Basic Predicate Logic.

For lists $\vec{A} = (A_1, A_2, \ldots, A_n)$ define $k(\vec{A}) \;:=\; (k(A_1), k(A_2), \ldots, k(A_n))$. Define $\vec{A} \;\vdash^k\; B$ by $k(\vec{A}) \;\vdash\; k(B)$. We define $\vdash^k A$ by $\top \vdash^k A$. We define $A \;\equiv^k\; B$ by $k(A) \;\equiv\; k(B)$.

Let BQC$^a$-E be the theory axiomatizable by the addition of schema $A^{aa} \;\vdash\; A$ to BQC-E. Over BQC$^a$-E letter $a$ fulfills a role of $\bot$ since $a \vdash A^{aa} \vdash A$ for all $A$. Let CQC$^a$-E be the theory axiomatizable by the addition of $a \vdash \bot$ to CQC-E.

**Proposition 5.1.** CQC$^a$*-E equals* BQC$^a$*-E.*

*Proof.* We only prove the less easy direction from $\mathrm{BQC}^a$-E. First, $a \vdash \top^a \equiv \bot^{aa} \vdash \bot$, so $a \equiv \bot$. Second, $(\top \to A) \wedge A^a \ \vdash\ \top^a \ \equiv\ a^{aa} \ \vdash\ a$, so $\top \to A \ \vdash\ A^{aa} \ \vdash\ A$. So $\mathrm{BQC}^a$-E proves intuitionistic logic IQC-E with $a \vdash \bot$ and schema $\neg\neg A \vdash A$. Thus also $\mathrm{CQC}^a$-E. □

Map $A \mapsto k(A)$ is a logical embedding of $\mathrm{CQC}^a$-E into BQC-E in a strong sense. We first show that $\vdash^k$ obeys all the axioms and rules of BQC-E of Section 2 plus the schema $A^{aa} \ \vdash^k\ A$. After that we give a simple proof that this completely axiomatizes $\vdash^k$.

**Proposition 5.2.** $A^{aa} \ \equiv^k\ A$.

*Proof.* By Proposition 4.1 we have $k(A^{aa}) \ =\ (k(A^a) \to k(a))^{aa} \ \equiv\ ((k(A) \to k(a))^{aa} \to \top^a)^{aa} \ \equiv\ ((k(A) \to \top^a)^{aa} \to \top^a)^{aa}$. With $a \vdash \top^a$ and K19 we have $(k(A) \to \top^a)^a \ \equiv\ (k(A)^a \vee \top^a)^a \ \equiv\ k(A)$. So $k(A^{aa}) \ \equiv\ (k(A)^a \ \to\ \top^a)^{aa} \ \equiv\ (k(A)^{aa} \vee \top^a)^{aa} \ \equiv\ k(A)^{aaaa} \ \equiv\ k(A)$. □

Next we show that $\vdash^k$ respects all axioms and rules of BQC-E. We may skip some trivially verifiable ones. Many 'bookkeeping' rules follow from $k(A \wedge B) \ \equiv\ k(A) \wedge k(B)$, which often allows us to ignore the $\vec{D}$ part in our proofs. We obviously have

$$A \vdash^k A \quad \text{and} \quad \frac{\vec{D} \ \vdash^k\ B}{\vec{D}, A \ \vdash^k\ B} \quad \text{and} \quad \frac{\vec{D}, A \ \vdash^k\ B \quad \vec{D}, B \ \vdash^k\ C}{\vec{D}, A \ \vdash^k\ C}$$

and

$$\frac{\vec{D}, A, B \ \vdash^k\ C}{\vec{D},\ A \wedge B \ \vdash^k\ C} \quad \text{and} \quad \frac{\vec{D},\ A \wedge B \ \vdash^k\ C}{\vec{D}, A, B \ \vdash^k\ C}$$

and

$$\frac{\vec{D} \ \vdash^k\ B}{\vec{D}[x/t] \ \vdash^k\ B[x/t]} \quad \text{no variable of term } t \text{ becomes bound}$$

We have $A \vdash^k \top$ since $k(A) \ \equiv\ k(A)^{aa} \ \vdash\ \top^{aa}$. We have $\bot \vdash^k A$ since $k(\bot) = \bot^{aa} \ \vdash\ k(A)^{aa} \ \equiv\ k(A)$.

**Proposition 5.3.** *We have*

$$\text{K38.} \ \frac{\vec{D} \ \vdash^k\ A \wedge B}{\vec{D} \ \vdash^k\ A} \quad \textit{and} \quad \frac{\vec{D} \ \vdash^k\ A \wedge B}{\vec{D} \ \vdash^k\ B} \quad \textit{and} \quad \frac{\vec{D} \ \vdash^k\ A \quad \vec{D} \ \vdash^k\ B}{\vec{D} \ \vdash^k\ A \wedge B}$$

$$\text{K39.} \ \frac{\vec{D},\ A \vee B \ \vdash^k\ C}{\vec{D}, A \ \vdash^k\ C} \quad \textit{and} \quad \frac{\vec{D},\ A \vee B \ \vdash^k\ C}{\vec{D}, B \ \vdash^k\ C} \quad \textit{and} \quad \frac{\vec{D}, A \ \vdash^k\ C \quad \vec{D}, B \ \vdash^k\ C}{\vec{D},\ A \vee B \ \vdash^k\ C}$$

*Proof.* Case K38. $k(C) \ \vdash\ k(A)$ plus $k(C) \ \vdash\ k(B)$ exactly when $k(C) \ \vdash\ k(A) \wedge k(B) \ \equiv\ k(A \wedge B)$.

Case K39. By K17 we have $k(A) \vee k(B) \ \ \vdash\ \ k(C)$ exactly when $k(A \vee B) \ \ \equiv\ (k(A) \vee k(B))^{aa} \ \vdash\ k(C)$. Use that $k(D) \wedge k(A \vee B) \ \equiv\ k((D \wedge A) \vee (D \wedge B))$, see K13 and K16. □

The following are all obvious:

$P(\mathbf{x}) \ \vdash^k\ \mathrm{E}\,\mathbf{x}$
$\mathrm{E}\,x \ \vdash^k\ x = x$ and $A \wedge x = y \ \vdash^k\ A[x/y]$ for atoms $A$
$\mathrm{E}\,f(\mathbf{x}) \ \vdash^k\ \mathrm{E}\,\mathbf{x}$
$\mathrm{E}\,\ell \ \vdash^k\ \bot$ (or schema $\mathrm{E}\,\ell \ \vdash^k\ A$)

**Proposition 5.4.** *We have*

K40. $\dfrac{\vec{D},\, A \wedge \mathrm{E}\, x \;\vdash^k\; B}{\vec{D},\, \exists x A \;\vdash^k\; B}$ *x not free in* $B, \vec{D}$ *and* $\dfrac{\vec{D},\, \exists x A \;\vdash^k\; B}{\vec{D},\, A \wedge \mathrm{E}\, x \;\vdash^k\; B}$

K41. $\dfrac{\vec{D},\, A \wedge B \wedge \mathrm{E}\, \mathbf{x} \;\vdash^k\; C}{\vec{D}, A \;\vdash^k\; \forall \mathbf{x}(B \to C)}$ *variables* $\mathbf{x}$ *not free in* $A, \vec{D}$

*Proof.* Case K40. First rule: $k(D)$, $k(A \wedge \mathrm{E}\, x) \;\vdash\; k(B)$ implies $k(D) \wedge k(A) \wedge \mathrm{E}\, x \;\equiv\; k(D) \wedge k(A) \wedge \top^{aa} \wedge \mathrm{E}\, x \;\equiv\; k(D) \wedge k(A) \wedge (\mathrm{E}\, x)^{aa} \wedge \mathrm{E}\, x \;\vdash\; k(B)$ implies $k(D)$, $\exists x\, k(A) \;\vdash\; k(B)$ implies $k(D)$, $k(\exists x A) \;\vdash\; k(B)^{aa} \;\equiv\; k(B)$. Second rule: It suffices to show $k(A \wedge \mathrm{E}\, x) \;\vdash\; k(\exists x A)$. We have $k(A) \wedge \mathrm{E}\, x \;\vdash\; \exists x\, k(A)$, so also $k(A) \wedge (\mathrm{E}\, x)^{aa} \;\vdash\; (\exists x\, k(A))^{aa} \;=\; k(\exists x A)$.

Case K41. From $k(A) \wedge k(B) \wedge k(\mathrm{E}\, \mathbf{x}) \;\vdash\; k(C)$ we get $k(A) \wedge k(B) \wedge \mathrm{E}\, \mathbf{x} \;\vdash\; k(C)$. So with Proposition 4.1 we have $k(A) \;\equiv\; k(A)^{aa} \;\vdash\; \forall \mathbf{x}\, (k(B) \to k(C))^{aa} \;\equiv\; k(\forall \mathbf{x}(B \to C))$. □

**Proposition 5.5.** *We have*

K42.

$\forall \mathbf{x}(A \to B) \;\vdash^k\; \forall \mathbf{x} y(A \to B)$ *y not free to the left of the entailment* *and*

$\forall \mathbf{x} y(A \to B) \;\vdash^k\; \forall \mathbf{x}((A \wedge \mathrm{E}\, y) \to B)$

K43.

$\forall \mathbf{x}(A \to B) \wedge \forall \mathbf{x}(B \to C) \;\vdash^k\; \forall \mathbf{x}(A \to C)$ *and*

$\forall \mathbf{x}(A \to B) \wedge \forall \mathbf{x}(A \to C) \;\vdash^k\; \forall \mathbf{x}(A \to (B \wedge C))$ *and*

$\forall \mathbf{x}(B \to A) \wedge \forall \mathbf{x}(C \to A) \;\vdash^k\; \forall \mathbf{x}((B \vee C) \to A)$ *and*

$\forall \mathbf{x} y(A \to B) \;\vdash^k\; \forall \mathbf{x}((A[y/t] \wedge \mathrm{E}\, t) \to B[y/t])$ *no variable of term t becomes bound in A or B* *and*

$\forall \mathbf{x} y(A \to B) \;\vdash^k\; \forall \mathbf{x}(\exists y A \to B)$ *y not free in B*

*Proof.* Case K42. Second item: With Proposition 4.1 the following suffices: $\forall \mathbf{x} y(k(A) \to k(B)) \;\vdash\; \forall \mathbf{x}((k(A) \wedge \mathrm{E}\, y) \to k(B)) \;\vdash\; \forall \mathbf{x}((k(A)^{aa} \wedge (\mathrm{E}\, y)^{aa}) \to k(B)^{aa}) \;\equiv\; \forall \mathbf{x}(k(A \wedge \mathrm{E}\, y) \to k(B))$.

Case K43. All cases are fairly straightforward. The third case for disjunction: With K20 and K11 we have $\forall \mathbf{x}\, (k(B \vee C) \to k(A))^{aa} \;\equiv\; \forall \mathbf{x}\, ((k(B) \vee k(C))^{aa} \to k(A))^{aa} \;\equiv\; \forall \mathbf{x}\, ((k(B) \vee k(C)) \to k(A))^{aa} \;\equiv\; \forall \mathbf{x}\, ((k(B) \to k(A)) \wedge (k(C) \to k(A)))^{aa} \;\equiv\; \forall \mathbf{x}\, (k(B) \to k(A))^{aa} \wedge \forall \mathbf{x}\, (k(C) \to k(A))^{aa}$. The fourth case follows from K27. For the fifth case for existential quantification the following suffices: $\forall \mathbf{x} y(k(A) \to k(B)) \;\vdash\; \forall \mathbf{x}(\exists y\, k(A) \to k(B)) \;\vdash\; \forall \mathbf{x}((\exists y\, k(A))^{aa} \to k(B)^{aa}) \;\equiv\; \forall \mathbf{x}(k(\exists y A) \to k(B))$. □

Starting from Proposition 5.2 we established that $\vdash^k$ is closed under all derivations of $\mathrm{BQC}^a$-E, so with Proposition 5.1 is also closed under all derivations of $\mathrm{CQC}^a$-E.

**Theorem 5.6.** *The derivations of* $\vdash^k$ *are exactly those of* $\mathrm{CQC}^a$*-E. There is a* $\mathrm{CQC}^a$*-E derivation of* $A \vdash B$ *from a collection of entailments* $A_i \;\vdash\; B_i$ *if and only if there is a* BQC*-E derivation of* $k(A) \;\vdash\; k(B)$ *from the collection of entailments* $k(A_i) \;\vdash\; k(B_i)$.

*Proof.* We established that $\mathrm{CQC}^a$-E derivations with entailments $A \vdash B$ imply BQC-E derivations with the corresponding $k(A) \;\vdash\; k(B)$.

Conversely, $\mathrm{CQC}^a$-E has classical predicate logic with $a \vdash \bot$, so over $\mathrm{CQC}^a$-E we have $A \;\equiv\; k(A) \;\equiv\; u(A)$ for all $A$. If there is a BQC-E derivation of $k(A) \;\vdash\; k(B)$ from a collection of entailments $k(A_i) \;\vdash\; k(B_i)$, then over extension $\mathrm{CQC}^a$-E $\supseteq$ BQC-E we have a derivation of $A \vdash B$ from collection of entailments $A_i \;\vdash\; B_i$. □

Note that $k(A) \equiv k(k(A)) \equiv k(u(A))$. So if $k(A) \vdash k(B)$ follows from a collection of entailments $A_i \vdash B_i$, then $k(A) \vdash k(B)$ also follows from collection of entailments $k(A_i) \vdash k(B_i)$.

In predicate logic one may prefer to work with the language $\mathcal{L}$ of BQC of [Ru98], without existence predicate $\mathrm{E}\,x$, with an inhabited intended domain, and with total functions. In the case of any predicate logic extending intuitionistic predicate logic IQC one also prefers universal quantification of form $\forall x A$. Such a predicate logic can be embedded into our language $\mathcal{L}$-E. Example: First, employ a straightforward translation $m$ definable by induction on the complexity of formulas, which essentially replaces $\forall x A$ by $\forall x(\top \to A)$. Second, perform the translation we mentioned immediately before Section 3, that is, replace entailments $A \vdash B$ by $A \wedge \mathrm{E}\,\mathbf{x} \vdash B$, where $\mathbf{x}$ is a list of all free variables that occur in $A$ or $B$. Third, add axioms $\vdash \exists x \top$ and $\mathrm{E}\,\mathbf{x} \vdash \mathrm{E}\,f(\mathbf{x})$ for all function symbols $f$, as indicated immediately before Section 3.

**Theorem 5.7.** *Let $A_1, \ldots, A_n, A$ be a list of formulas over* CQC*, and $\mathbf{x}$ be all free variables that occur in the list. Set propositional letter $a$ equal to $\bot$. Then $A_1, \ldots, A_n \vdash A$ in* CQC *if and only if $k(m(A_1)), \ldots, k(m(A_n)), \neg\neg\,\mathrm{E}\,\mathbf{x} \vdash k(m(A))$ over $\ell$-free* BQC *-E$\cup\{\vdash \exists x \top\} \cup \{\mathrm{E}\,\mathbf{x} \vdash \mathrm{E}\,f(\mathbf{x})\}_f$.*

*Proof.* Over BQC-E we have equivalences $\exists x A \equiv \exists x(\mathrm{E}\,x \wedge A)$ and $\forall \mathbf{x}(A \to B) \equiv \forall \mathbf{x}((A \wedge \mathrm{E}\,\mathbf{x}) \to B)$. So the translation is equivalent to a standard relativization of quantifiers, and $\vdash \exists x \top$ guarantees that the intended domain is inhabited. Clearly $m(B \wedge C) \equiv m(B) \wedge m(C)$ and $k(B \wedge C) \equiv k(B) \wedge k(C)$. Finally, apply Proposition 2.1. □

The BQC of [Ru98] has no existence predicate $\mathrm{E}\,x$ or constant symbol $\ell$, intended domains have elements, and all functions are total. Entailments $A \vdash B$ of BQC correspond with entailments $A \wedge \mathrm{E}\,\mathbf{x} \vdash B$ of $\ell$-free BQC-E $\cup \{\vdash \exists x \top\} \cup \{\mathrm{E}\,\mathbf{x}_i \vdash \mathrm{E}\,f(\mathbf{x})\}_f$. So

**Theorem 5.8.** *Let $A_1, \ldots, A_n, A$ be a list of formulas over* CQC*. Set propositional letter $a$ equal to $\bot$. Then*

> $A_1, \ldots, A_n \vdash A$ *in* CQC
> *if and only if*
> $k(m(A_1)), \ldots, k(m(A_n)), \neg\neg\top \vdash k(m(A))$ *in* BQC

*Condition $\neg\neg\top$ is redundant when $n > 0$.*

**Corollary 5.9** (Generalized Glivenko)**.** *Let $A$ and $B$ be formulas without any occurrence of form $\forall \mathbf{x}$ with non-empty $\mathbf{x}$. Then*

> $A \vdash B$ *in* CQC
> *if and only if*
> $\neg\neg A \vdash \neg\neg B$ *in* BQC

# References


[AAR16] Majid Alizadeh, Mohammad Ardeshir, Wim Ruitenburg. *Boolean algebras in Visser algebras*, Notre Dame Journal of Formal Logic 57 (2016), pp. 141–150.

[ArRu98] Mohammad Ardeshir, Wim Ruitenburg. *Basic propositional calculus I*, Mathematical Logic Quarterly 44 (1998), pp. 317–343.

[ArRu20] Mohammad Ardeshir, Wim Ruitenburg. *A constructive interpretation of the logical constants*, submitted.

[vdB19] Benno van den Berg. *A Kuroda-style j-translation*, Archive for Mathematical Logic 58 (2019), pp. 627–634.

[BrRi14] Chad E. Brown, Christine Rizkallah. *Glivenko and Kuroda for simple type theory*, The Journal of Symbolic Logic 79 (2014), pp. 485–495.

[Dr79] A.G. Dragalin. **Mathematical Intuitionism, Introduction to Proof Theory**, Translations[1] of Mathematical Monographs 67, American Mathematical Society 1988.

[FMS79] Michael Fourman, Chris Mulvey, Dana Scott (editors). **Applications of Sheaves**, Lecture Notes in Mathematics 753, Springer-Verlag, 1979.

[Gl29] Valery Glivenko. *Sur quelques points de la logique de M. Brouwer*, Bulletins de la classe des sciences 15 (1929), pp. 183–188.

[Hei67] Jean van Heijenoort. **From Frege to Gödel, A Source Book in Mathematical Logic, 1879–1931**, Harvard University Press, 1967.

[Ko25] A.N. Kolmogorov. *O printsipe tertium non datur*, Mathematičeskiĭ Sbornik 32 (1925), pp. 646–667. English translation *On the principle of excluded middle*, in: [Hei67, pp. 414–437].

[Ku51] Sigekatu Kuroda. *Intuitionistische Untersuchungen der formalistischen Logik*, Nagoya Mathematical Journal 2 (1951), pp. 35–47.

[Ru98] Wim Ruitenburg. *Basic predicate calculus*, Notre Dame Journal of Formal Logic 39 (1998), no. 1, pp. 18–46.

[Sc79] D.S. Scott. *Identity and existence in intuitionistic logic*, in: [FMS79, pp. 660–696].

[TrvD88] A.S. Troelstra, D. van Dalen. **Constructivism in Mathematics, an Introduction, Volume 1**, Studies in Logic and the Foundations of Mathematics 121, North-Holland, 1988.

[Vi81] Albert Visser. *A propositional logic with explicit fixed points*, Studia Logica 40 (1981), pp. 155–175.


[1] The original in Russian is from 1979.