A special logic for transitive Kripke models

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1 New BQC-2023 (or BQC-23)

The logical symbols are $\top$, $\bot$, $A \wedge B$, $A \vee B$, $\exists x A$, and $\forall \mathbf{x}(A \to B)$.

A1. $$\frac{\vec{D}, A, A \Rightarrow B}{\vec{D}, A \Rightarrow B} \qquad \frac{\vec{D}, A, B \Rightarrow C}{\vec{D}, B, A \Rightarrow C} \qquad \frac{\vec{D}, A \Rightarrow B}{A, \vec{D} \Rightarrow B}$$

So $\vec{D}$ in $\vec{D} \Rightarrow B$ is essentially a finite set of formulas.

A2. $$\vec{D}, A \Rightarrow A \qquad \frac{\vec{D} \Rightarrow B}{\vec{D}, A \Rightarrow B} \qquad \frac{\vec{D} \Rightarrow B \quad \vec{D}, B \Rightarrow C}{\vec{D} \Rightarrow C}$$

The 'weakening' second rule of A2 makes that finite $\vec{D}$ are not necessary in sequent axioms (below we don't bother to leave such $\vec{D}$ out).

A3. $$\frac{\vec{D}, A, B \Rightarrow C}{\vec{D}, A \wedge B \Rightarrow C} \qquad \frac{\vec{D}, A \wedge B \Rightarrow C}{\vec{D}, A, B \Rightarrow C}$$

So $\vec{D} \Rightarrow B$ and $\bigwedge \vec{D} \Rightarrow B$ are essentially the same for all $\vec{D}$ and B. We may write $\vec{D}$ for $\bigwedge \vec{D}$ whenever convenient.

A4. $$\frac{\vec{D} \Rightarrow A \wedge B}{\vec{D} \Rightarrow A} \qquad \frac{\vec{D} \Rightarrow A \wedge B}{\vec{D} \Rightarrow B} \qquad \frac{\vec{D} \Rightarrow A \quad \vec{D} \Rightarrow B}{\vec{D} \Rightarrow A \wedge B}$$

A5. $\vec{D} \Rightarrow \top$

A6. $$\frac{\vec{D}, A \vee B \Rightarrow C}{\vec{D}, A \Rightarrow C} \qquad \frac{\vec{D}, A \vee B \Rightarrow C}{\vec{D}, B \Rightarrow C} \qquad \frac{\vec{D}, A \Rightarrow C \quad \vec{D}, B \Rightarrow C}{\vec{D}, A \vee B \Rightarrow C}$$

A7. $\vec{D}, \bot \Rightarrow B$

A8. $\vec{D} \Rightarrow x = x \qquad \vec{D}, A, x = y \Rightarrow A[x/y]$ for atoms A

A9. $$\frac{\vec{D} \Rightarrow B}{\vec{D}[x/t] \Rightarrow B[x/t]}$$ no variable of term t becomes bound

A10. $$\frac{\vec{D}, A \Rightarrow B}{\vec{D}, \exists x A \Rightarrow B}$$ x not free in $B, \vec{D}$ $$\frac{\vec{D}, \exists x A \Rightarrow B}{\vec{D}, A \Rightarrow B}$$

The fragment above with restriction to sequents $\vec{D} \Rightarrow B$ of formulas built from the atoms using only $\wedge$, $\vee$, and $\exists$, is the well-known finite geometric logic.

Implication $A \to B$ is short for the special case $\forall(A \to B)$. Negation $\neg A$ is defined by $A \to \bot$.

A11. $\dfrac{\vec{D}, A \ \Rightarrow \ B}{\vec{D} \ \Rightarrow \ \forall \mathbf{x}(A \to B)}$ variables $\mathbf{x}$ not free in $\vec{D}$

A12. $\vec{D}, \forall \mathbf{x}(A \to B) \ \Rightarrow \ \forall \mathbf{x}y(A \to B)$ y not free left of the sequent arrow

A13. $\vec{D}, \forall \mathbf{x}y(A \to B) \ \Rightarrow \ \forall \mathbf{x}(A \to B)$

A14. $\vec{D}, \forall \mathbf{x}(A \to B), \forall \mathbf{x}(B \to C) \ \Rightarrow \ \forall \mathbf{x}(A \to C)$

A15. $\vec{D}, \forall \mathbf{x}(A \to B), \forall \mathbf{x}(A \to C) \ \Rightarrow \ \forall \mathbf{x}(A \to (B \wedge C))$

A16. $\vec{D}, \forall \mathbf{x}(B \to A), \forall \mathbf{x}(C \to A) \ \Rightarrow \ \forall \mathbf{x}((B \vee C) \to A)$

A17. $\vec{D}, \forall \mathbf{x}y(A \to B) \ \Rightarrow \ \forall \mathbf{x}(\exists y A \to B)$ y not free in B

This completes the axiomatization of BQC-2023.

Intuitionistic Predicate Logic IQC-2023 is definable by the addition of schema $\top \to A \ \Rightarrow \ A$, which allows one to derive modus ponens $A \wedge (A \to B) \Rightarrow B$. Classical Predicate Logic CQC-2023 is definable by adding schemas $\top \to A \ \Rightarrow A$ plus Excluded Middle $\Rightarrow \ A \vee \neg A$ or, alternatively, by adding the single schema of double negation elimination $\neg\neg A \ \Rightarrow \ A$.

Proposition 1.1. *A list of derivable entailments over* BQC-*2023.*

B1. $\vdash \ A \wedge (B \vee C) \ \Leftrightarrow \ (A \wedge B) \vee (A \wedge C)$

B2. $\vdash \ A \wedge \exists x B \ \Leftrightarrow \ \exists x(A \wedge B)$ *x not free in A*

B3. $\vdash \ \top \to \bot \ \Rightarrow \ \forall \mathbf{x}(A \to B)$

B4. $\vdash \ \forall \mathbf{x}(A \to B) \ \Leftrightarrow \ (\exists \mathbf{x} A \to B)$ *no x in $\mathbf{x}$ free in B*

Proposition 1.2 (Formula substitution)**.** *Let $\mathcal{L}$ be a language, p be a new propositional letter, $C[p] \in \mathcal{L}[p]$, and $A, B \in \mathcal{L}$. Then* BQC-*2023 proves*

$$\vec{D}, A \Rightarrow B, \ \ \vec{D}, B \Rightarrow A \ \ \vdash \ \ \vec{D}, C[A] \Rightarrow C[B]$$

where no variable that occurs free in both $\vec{D}$ and in A, B becomes bound after substitution of A and B in $C[p]$.

Renaming bound variables.

Proposition 1.3. *Let C be a formula in which the variables x and y don't occur free, and neither x nor y becomes bound after substitutions $C[z/x]$ or $C[z/y]$. Then* BQC-*2023 proves $D[\exists x C[z/x]] \Leftrightarrow D[\exists y C[z/y]]$, for all contexts $D[p]$.*

Proposition 1.4. *Let A and B be formulas in which the variables in $\mathbf{x}$ and $\mathbf{y}$ don't occur free, and where no variable in $\mathbf{x}$ or $\mathbf{y}$ becomes bound after substitutions $A[\mathbf{z}/\mathbf{x}]$, $B[\mathbf{z}/\mathbf{x}]$, $A[\mathbf{z}/\mathbf{y}]$, or $B[\mathbf{z}/\mathbf{y}]$. Lists $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ have the same length. Then* BQC-*2023 proves $D[\forall \mathbf{x}(A[\mathbf{z}/\mathbf{x}] \to B[\mathbf{z}/\mathbf{x}])] \Leftrightarrow D[\forall \mathbf{y}(A[\mathbf{z}/\mathbf{y}] \to B[\mathbf{z}/\mathbf{y}])]$, for all contexts $D[p]$.*

1.1 Functional Well-formed Theories

BQC-2023 is the theory of transitive Kripke models similar to how intuitionistic predicate logic IQC-2023 is the theory of reflexive transitive Kripke models. Theories over transitive Kripke models essentially satisfy the extra properties of being functional and well-formed.

Theories are sets of rules generated by sets BQC-2023 $\cup\, \Gamma$, where Γ is a set of rule axioms R of form

$$R \;:=\; \frac{\vec{D}_1 \Rightarrow B_1 \;\ldots\; \vec{D}_n \Rightarrow B_n}{\vec{D}_0 \Rightarrow B_0}$$

$\Gamma \vdash R$ if and only if

$$\Gamma \cup \{\vec{D}_1 \Rightarrow B_1,\; \ldots,\; \vec{D}_n \Rightarrow B_n\} \;\vdash\; \vec{D}_0 \Rightarrow B_0$$

Define rule $A \times R$ by

$$A \times R \;:=\; \frac{\vec{D}_1, A \Rightarrow B_1 \;\ldots\; \vec{D}_n, A \Rightarrow B_n}{\vec{D}_0, A \Rightarrow B_0}$$

Proposition 1.5. *Derivable entailments over* BQC-*2023.*

B5. $\vdash\; \bot \times R$

B6. $\top \times R \;\dashv\vdash\; R$

B7. $A \times (B \times R) \;\dashv\vdash\; (A \wedge B) \times R$

B8. If variables $\mathbf{z}$ *are not free in rule* R*, then* $A \times R \;\dashv\vdash\; \exists \mathbf{z} A \times R$

Set of rules Γ is *functional* if for all rules R and formulas A with only 'new' variables, we have $\Gamma \vdash R$ implies $\Gamma \;\vdash\; A \times R$.

Proposition 1.6. *Let* $\Gamma \cup \{R\}$ *be a set of rules such that* $\Gamma \vdash R$*, and* A *be a sentence. Then* $A \times \Gamma \;\vdash\; A \times R$*.*

Proposition 1.7. *A theory* Δ *is functional if and only if* Δ *has a functional axiomatization.*

Given a rule

$$R := \frac{\vec{D}_1 \Rightarrow B_1 \ \ldots \ \vec{D}_n \Rightarrow B_n}{\vec{D}_0 \Rightarrow B_0}$$

and list of variables $\mathbf{x}$, we write $\int_{\mathbf{x}} R$ for sequent

$$\forall \mathbf{x}(\bigwedge \vec{D}_1 \to B_1), \ldots, \forall \mathbf{x}(\bigwedge \vec{D}_n \to B_n) \Rightarrow \forall \mathbf{x}(\bigwedge \vec{D}_0 \to B_0)$$

Proposition 1.8. *Let R be a rule and $\mathbf{xy}$ be a list of variables. Then $\int_{\mathbf{x}} R \vdash \int_{\mathbf{xy}} R$. If none of the $\mathbf{y}$ are free in the (numerator) suppositions of R, then $\int_{\mathbf{xy}} R \vdash \int_{\mathbf{x}} R$.*

We write[1] $\int_R S$ for $\int_{\mathbf{x}} S$ if $\mathbf{x}$ equals the free variables of rule R. So $\int_R R \dashv\vdash \int_{\mathbf{x}} R$ whenever $\mathbf{x}$ includes all free variables in the (numerator) suppositions of R.

Proposition 1.9. *Let $\Gamma \cup \{R\}$ be a set of rules such that $\Gamma \vdash R$. Then $\int \Gamma \vdash \int_R R$.*

Set of rules Γ is *well-formed* if for all rules R and formulas A with only 'new' variables, we have $\Gamma \vdash R$ implies $\Gamma \vdash \int_R (A \times R)$.

Proposition 1.10. *Derivable entailments over* BQC-*2023.*

B9. $\int_{\mathbf{x}} R \dashv\vdash \int_{\mathbf{x}} (\top \times R)$

B10. $\int_{\mathbf{x}} R \vdash A \times \int_{\mathbf{x}} R$

B11. $\int_{\mathbf{x}} (A \times R) \vdash A \times \int_{\mathbf{x}} R$ *whenever the variables* $\mathbf{x}$ *aren't free in* A

Proposition 1.11. *A theory Δ is well-formed if and only if Δ has a well-formed axiomatization.*

[1] A derivate $(\vec{D} \Rightarrow B)'$ of 'differentiable' sequents $\vec{D} \Rightarrow B$ exists satisfying $(\int_R R)' \dashv\vdash R$, where $\vec{D} \Rightarrow B$ is *differentiable* when $\vec{D}, B$ is a list of universal implication sentences.

2 Transitive Kripke Models for BQC-2023

A Kripke model $\mathfrak{A}$ consists of the following components.

First, a structure $(W, \sqsubset)$ of a non-empty set of worlds or nodes W with transitive relation $\sqsubset$. We write $\sqsubseteq$ for the reflexive closure of $\sqsubset$.

Second, for each $k \in W$ there is a classical model $\mathfrak{A}_k$, and for all pairs $k \sqsubseteq m$ there is an algebraic morphism (preserving atoms) $\upharpoonright^k_m : \mathfrak{A}_k \to \mathfrak{A}_m$ such that $\upharpoonright^k_k$ is the identity for all k, and $\upharpoonright^m_n \upharpoonright^k_m = \upharpoonright^k_n$ for all $k \sqsubseteq m \sqsubseteq n$.

Given a Kripke model $\mathfrak{A}$ over $\mathcal{L}$ with node $k \in W$, classical model $\mathfrak{A}_k$ has domain A_k, and language $\mathcal{L}(A_k)$ with new constant symbols. Define classical truth interpretation $\mathfrak{A}_k \models B$ for sentences $B \in \mathcal{L}(A_k)$ as usual. Function $\upharpoonright^k_m : A_k \to A_m$ implies a formula translation $B \mapsto B^k_m$ from $\mathcal{L}(A_k)$ to $\mathcal{L}(A_m)$. Similarly for rules $R \mapsto R^k_m$.

Forcing $(\mathfrak{A}, k) \Vdash B$ for sentences $B \in \mathcal{L}(A_k)$ is inductively definable by:

$(\mathfrak{A}, k) \Vdash B$ if and only if $\mathfrak{A}_k \models B$, for all atomic sentences $B \in \mathcal{L}(A_k)$
$(\mathfrak{A}, k) \Vdash B \wedge C$ if and only if $(\mathfrak{A}, k) \Vdash B$ and $(\mathfrak{A}, k) \Vdash C$
$(\mathfrak{A}, k) \Vdash B \vee C$ if and only if $(\mathfrak{A}, k) \Vdash B$ or $(\mathfrak{A}, k) \Vdash C$
$(\mathfrak{A}, k) \Vdash \exists x C$ if and only if there is $c \in A_k$ such that $(\mathfrak{A}, k) \Vdash C[x/c]$
$(\mathfrak{A}, k) \Vdash \forall \mathbf{x}(B \to C)$ if and only if for all $m \sqsupset k$ and $\mathbf{c} \in A_m$ we have $(\mathfrak{A}, m) \Vdash B^k_m[\mathbf{x}/\mathbf{c}]$ implies $(\mathfrak{A}, m) \Vdash C^k_m[\mathbf{x}/\mathbf{c}]$

Proposition 2.1 (Persistence of forcing for sentences)**.** *Let $k \sqsubseteq m$ be nodes of a transitive Kripke model $\mathfrak{A}$, and B be a sentence over $\mathcal{L}(A_k)$. Then $(\mathfrak{A}, k) \Vdash B$ implies $(\mathfrak{A}, m) \Vdash B^k_m$.*

Write $k \Vdash$ for $(\mathfrak{A}, k) \Vdash$ if the Kripke model $\mathfrak{A}$ is clear from the context.

With Proposition 2.1 we extend forcing from sentences to formulas $B \in \mathcal{L}(A_k)$ with all free variables among $\mathbf{x}$ by

$$k \Vdash B \text{ if and only if for all } m \sqsupseteq k \text{ and } \mathbf{c} \in A_m \text{ we have } m \Vdash B^k_m[\mathbf{x}/\mathbf{c}]$$

Similarly for lists of formulas $\vec{D}$. The empty list is always forced.

Extend forcing to all sequents by

$k \Vdash (\vec{D} \Rightarrow B)$ if and only if for all $m \sqsupseteq k$ and $\mathbf{c} \in A_m$ we have $m \Vdash \vec{D}^k_m[\mathbf{x}/\mathbf{c}]$ implies $m \Vdash B^k_m[\mathbf{x}/\mathbf{c}]$

So $k \Vdash B$ if and only if $k \Vdash (\Rightarrow B)$.

Proposition 2.2 (Persistence of forcing for sequents)**.** *Let $k \sqsubseteq m$ be nodes of a transitive Kripke model $\mathfrak{A}$, and $\vec{D} \Rightarrow B$ be a sequent over $\mathcal{L}(A_k)$. Then $k \Vdash (\vec{D} \Rightarrow B)$ implies $m \Vdash (\vec{D} \Rightarrow B)^k_m$.*

Let R be rule

$$\frac{\vec{D}_1 \;\Rightarrow\; B_1 \;\ldots\; \vec{D}_n \;\Rightarrow\; B_n}{\vec{D}_0 \;\Rightarrow\; B_0}$$

Define

$k \Vdash R$ if and only if for all $m \sqsupseteq k$ we have $m \Vdash (\vec{D}_i \Rightarrow B_i)^k_m$ for all $i > 0$ implies $m \Vdash (\vec{D}_0 \Rightarrow B_0)^k_m$

With Proposition 2.2 we have $k \Vdash (\vec{D} \Rightarrow B)$ as a sequent exactly when $k \Vdash (\vec{D} \Rightarrow B)$ as a rule with empty list of suppositions.

Proposition 2.3 (Persistence of forcing for rules)**.** *Let $k \sqsubseteq m$ be nodes of a transitive Kripke model $\mathfrak{A}$, and R be a rule over $\mathcal{L}(A_k)$. Then $k \Vdash R$ implies $m \Vdash R^k_m$.*

Forcing of universal implication sentences corresponds with sequent forcing as follows.

Proposition 2.4. *Let k be a node of a transitive Kripke model $\mathfrak{A}$, and $\forall \mathbf{x}(B \to C)$ be a sentence over $\mathcal{L}(A_k)$. Then $k \Vdash \forall \mathbf{x}(B \to C)$ if and only if $n \Vdash (B^k_n \Rightarrow C^k_n)$ for all $n \sqsupset k$.*

Proof. Both statements are equivalent to

For all $n \sqsupset k$ and $\mathbf{d} \in A_n$ we have $(\mathfrak{A}, n) \Vdash B^k_n[\mathbf{x}/\mathbf{d}]$ implies $(\mathfrak{A}, n) \Vdash C^k_n[\mathbf{x}/\mathbf{d}]$

□

There is a generalization of Proposition 2.4 to formulas.

Proposition 2.5 (Soundness)**.** *Let* $\Gamma \cup \{R\}$ *be a set of rules. Then* $\Gamma \vdash R$ *implies* $\Gamma \Vdash R$*.*

For each node k of a transitive Kripke model $\mathfrak{A}$ we define set of rules $\mathrm{Th}(\mathfrak{A}, k)$ over $\mathcal{L}(A_k)$ by

$$\mathrm{Th}(\mathfrak{A}, k) \ := \ \{R \in \mathcal{L}(A_k) \ | \ k \Vdash R\}$$

Proposition 2.6. *Let* k *be a node of transitive Kripke model* $\mathfrak{A}$*. Then* $\mathrm{Th}(\mathfrak{A}, k)$ *is a functional well-formed theory.*

Proposition 2.7 (Completeness)**.** *Let* $\Gamma \cup \{R\}$ *be a set of rules, and* Γ *be functional and well-formed. Then* $\Gamma \Vdash R$ *implies* $\Gamma \vdash R$*.*