A Constructive Interpretation of the Logical Constants

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Abstract

Heyting's intuitionistic predicate logic describes very general regularities observed in constructive mathematics. The intended meaning of the logical constants is clarified through Heyting's proof interpretation. A re-evaluation of proof interpretation and predicate logic leads to the new constructive Basic logic properly contained in intuitionistic logic. We develop logic and interpretation simultaneously by an axiomatic approach. Basic logic appears to be complete. A brief historical overview shows that our insights are not all new.

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1 Introduction

Many of our insights that lead to Basic logic, a proper subsystem of intuitionistic logic, are similar to insights of scholars, often constructivists, who discuss aspects of constructive mathematics and logic along lines similar to Heyting's intuitionistic logic and proof interpretation. Their contributions include significant questions about, proposed modifications to, or improved clarifications of Heyting's proof interpretation. Some main concerns are the use of hypothetical statements and the interpretation of implication. Intuitionistic logic itself as *the* constructive predicate logic isn't challenged[1].

We conclude that intuitionistic logic itself needs to change before the logical constants can be constructively justified. Our principal method is (1) a simultaneous development of a constructive proof interpretation and a constructive logic by (2) axiomatic methods. The result is (3) a very general axiom system of constructions and proofs. A moderate axiom system of constructions and proofs already suffices to obtain a constructive Basic logic which appears complete by using weak counterexamples in a sense by which intuitionistic logic may be considered complete. Completeness of the axiom system for Basic

[1] In the 1930s Ingebrigt Johansson challenged the use of Ex Falso, which led to the development of Minimal Logic. In the 1940s George Griss argued that negation, and hence a whole class of implications, should be banished from intuitionism. The differences with intuitionistic predicate logic, maybe after restriction to a sublanguage, are not significant. Hans Freudenthal's observations in the 1930s didn't from his side lead to an alternate logic for intuitionism.

logic may not extend to completeness of the axiom system for constructions and proofs. Instead, its generality may be the basis for interesting unintended interpretations. When compared with intuitionistic logic, the main changes with Basic logic are a weakening of the rules for implication, and a corresponding weakening of the rules for universal quantification. The main reason for the need for these changes is that in the context of very general theories we must allow for (4) an expanding 'universe' of constructions and proofs. The constructively unavoidable possibility of the discovery of further methods of proof is the principal concern.

We don't challenge the mathematical value of intuitionistic logic (or of classical logic, for that matter). We can still ask the question: What is the place of intuitionistic logic if it isn't the logic of constructive mathematics (it is the internal logic of category theory and topos theory)? Another open question is: What are the mathematical strengths of theories over Basic logic?

The proposition logical fragment of Basic logic is due to Albert Visser, see [Vi81]. An early discussion about a new proof interpretation is in [Ru93]. The predicate logic BQC with completeness theorem is in [Ru98].

1.1 Outline

Section 2 Constructive Mathematics
We consider the three most recognized schools of constructive mathematics. Additionally, we mention the connection between intuitionistic logic and category theory, which exists without requiring a constructive interpretation.

Section 3 Mathematical Logic as Applied Mathematics
We sketch the origins of intuitionistic predicate logic. Its axioms should reflect observed regularities in the use of language by constructivists. From a constructivist point of view one may never be able to prove whether the axiom system is complete, as further methods of proof may be discovered. Weak counterexamples offer arguments in favor of completeness.

Section 4 Proof Interpretations for Intuitionistic Logic
In reply to concerns about the constructive validity of the axiom system of intuitionistic predicate logic, Heyting developed a proof interpretation, of which we quote a version dated 1988. Despite the practical acceptance of intuitionistic predicate logic, there is no generally accepted constructive interpretation for its logical constants. The main concern involves implication and quantification over a totality of constructions. This concern doesn't apply to the well known connection between intuitionistic logic and category theory.

Section 5 A General Theory of Constructions and Proofs
We precede constructive predicate logic by a sufficiently detailed proof interpretation so that its most obvious existence and closure rules obtain all of Basic predicate logic. We neither pretend to have nor need a complete theory of proofs to obtain this result. With respect to the usual sublanguage without existence predicate $\mathrm{E}\,x$ we get the Basic predicate logic BQC, for which there is a Kripke model completeness theorem in [Ru98]. Weak counterexamples that respect the possibility of the discovery of further methods of proof offer constructive arguments in favor of completeness.

Section 6 Conclusion and Remarks
We perceive that constructive type theories along the lines of Martin-Löf appear to restrict what counts as permitted constructions and proofs in such a way that modus ponens holds. The restrictions don't agree with the constructively recognized possibility of an open ended universe of proofs.

2 Constructive Mathematics

Following Bridges and Richman on [BrRi87, page 1] there are, from a historical perspective, three major schools of constructive mathematics: Brouwer's intuitionism, Markov's constructivism, and Bishop's constructive mathematics. We consider one other source of constructive mathematics and logic for which the term 'school' may not apply, category theory in general, and topos theory in particular. All four are distinct, and none is contained in all others[2]. This contrasts with Section 3: All four agree on Heyting's intuitionistic logic as their constructive predicate logic.

As an aside, note that the poorly chosen name 'Basic logic' has precedences in names like 'Intuitionism' and 'Topos'.

2.1 Brouwer, Markov, Bishop

The first fully constructive mathematics and philosophy began with L.E.J. Brouwer's 1907 PhD thesis [Br07, He75]. Brouwer condemned a logical foundation of mathematics independent of proper a priori human mental concepts. For otherwise one builds a linguistic structure definitely distinct from mathematics, see [Br07, pages 179–180]. Brouwer's PhD student Arend Heyting on [He78, page 7] wrote that

> Brouwer became the founder and defender of the special form of constructive mathematics which he called intuitionism, a denomination which was the cause of much misunderstanding.

Brouwer's intuitionism includes what are called choice sequences, a notion which he considered intuitively clear. Nonetheless he struggled at times in dealing with them, see [He78, page 11]. Brouwer's intuitionism is not consistent with classical mathematics, see Theorem 3.6 on [BrRi87, page 115].

On [Ku06, page 559] Boris A. Kushner writes

> After World War II Markov's interests turned to axiomatic set theory, mathematical logic, and the foundations of mathematics. He founded the Russian school of constructive mathematics in the late 1940s and early 1950s. But in private conversations Markov often said that he had nurtured constructivist convictions for a very long time, in fact, long before the war.

The objects in A.A. Markov's constructive mathematics may be considered as words over finite alphabets. Some Markov constructivists contemplated broader possibilities. For example, Vladik Kreinovich offers a list of 5 Main Challenges on [Kr16, page 218], of which the first two are:

- The need to extend constructive mathematics to more complex mathematical objects.
- To be useful for data processing, algorithms must be able to handle possibly non-constructive data.

On the one hand Markov didn't recognize Brouwer's choice sequences in his own philosophy. On the other he accepted a principle now called Markov's Principle. Markov's constructivism is not consistent with classical mathematics, see Theorem 6.5 on [BrRi87, page 69].

Arguably the first substantial body of constructive mathematics is in [Bi67] where, on [Bi67, page ix] Errett Bishop wrote that

[2] Gödel mentioned on [Go38, Go95, pages 88–89] the haziness of the concept 'constructive'.

> This development is carried through with an absolute minimum of philosophical prejudice concerning the nature of constructive mathematics. There are no dogmas to which we must conform. Our program is simple: to give numerical meaning to as much as possible of classical abstract analysis.

Bishop's constructivism includes choice principles extending Countable Choice. His constructivism is consistent with classical mathematics. Bishop's book is restricted to constructive analysis. For constructive algebra, see [MRR88].

2.2 Topos Theory

In the early 1960s appeared theorems about categories which, as Johnstone wrote on [Jo77, page xii],

> ... paved the way for a truly autonomous development of category theory as a foundation for mathematics.

The development of principal interest to us began with F.W. Lawvere's [La64], titled *An elementary theory of the category of sets*. Once Lawvere turned his attention to Grothendieck toposes as generalized set theories, a new theory of the category of sets evolved called (elementary) topos theory. A name which Johnstone on page [Jo77, page xii], in an earlier context, called "... slightly unfortunate ...". Topos theory is a major part of an autonomous development of category theory as a foundation for mathematics. What makes topos theory also constructive is, that the mathematics done on the inside is in general restricted to the rules of intuitionistic logic shared by the three 'schools' mentioned before.

Internal topos mathematics is consistent with classical mathematics, but doesn't axiomatize any choice principles, not even Countable Choice. An additional natural number object also has an elementary category theoretic axiomatization, as shown by P. Freyd in [Fr72]. Internal topos mathematics allows for a so-called truth value object with which one can, using set terminology, freely construct power sets of sets. Such liberal set constructions are not part of Brouwer's intuitionism.

3 Mathematical Logic as Applied Mathematics

The original motivation for intuitionistic predicate logic appears to have been the capture of observed regularities in intuitionistic mathematics.

Brouwer and Bishop approached constructive mathematics primarily, though certainly not completely, without considering general rules of (constructive) logic. Markov's original approach was primarily through computability. All had interest in constructive logic, but as a secondary concern. Topos theory internal logic happens to be the intuitionistic one, but as a consequence of category theoretic considerations.

Heyting introduced intuitionistic 'constructive' logic in the years 1927–1930. He already studied axiomatic theories of intuitionistic geometry and algebra, which use abstract general hypotheticals.

Heyting wrote on [He78, page 8] that

> Logic can be considered in different ways. As a study of regularities in language it is an experimental science which, like any such science, needs mathematical notions; therefore it belongs to applied mathematics.

The realization that mathematical logic can be seen as applied mathematics dates back to the 1930s. A constructivist can consider classical mathematical logic, but the linguistic structure is definitely distinct from (constructive) mathematics, for classical logic includes the principle of Excluded Middle, which constructivists deny as a general principle.

Consider the following two questions about the development of logical systems for classical mathematics. It suffices to choose $A \wedge B$, $\neg A$, and $\forall x A$ as the logical constants for classical predicate logic. First, did a classical mathematician in 1927 have with predicate logic a 'good' formal language in the sense of expressive power? One may reply yes since with the right collection of atomic formulas and choice of theory there is strong expressive power. Second, did a classical mathematician in 1927 have a complete set of axioms and rules? This was answered in the affirmative by Gödel's Completeness Theorem of 1929–1930. If one wants to defend a logic as the constructive one, then at least this logic must have substantial richness to express constructive mathematics. Therefore we ask the same questions about intuitionistic logic as we did for classical logic.

Already in 1923 Brouwer showed ([Br24] is a German translation of [Br23]) that $\neg A$ is equivalent to $\neg\neg\neg A$. In 1927 the Wiskundig Genootschap, the Dutch 'Mathematical Society' posted a prize question about a formalization of Brouwer's intuitionistic mathematics, essentially including the problem of formalizing an intuitionistic predicate logic (the full problem statement was more involved). To this Heyting wrote an essay for which he was awarded the prize the following year. An expanded version of the essay appeared in [He30a, He30b, He30c]. An earlier partial version due to A.N. Kolmogorov in [Ko25] used a more restricted language, and for that restricted language he didn't include all rules implied by Heyting's version. In the introduction to the English translation of [Ko25], Hao Wang wrote on [Hei67, page 414] that

> To a large extent, this paper anticipated not only Heyting's formalization of intuitionistic logic, but also results on the translatability of classical mathematics into intuitionistic mathematics.

How about the two questions? Heyting chose a formal language with a collection of logical operators equivalent to $\top$, $\perp$, $A \wedge B$, $A \vee B$, $A \to B$, $\neg A$, $\forall x A$, and $\exists x A$. The axioms and rules are a proper subset of the axioms and rules known for classical logic (we ignore some axioms about equality). Is Heyting's language a 'good' language because of its expressive power? The material in [Bi67] and [MRR88] forms an incomplete list that can be converted into theories and propositions over Heyting's predicate logic. This is a positive answer to the first question.

Second question: Did Heyting have a complete set of axioms and rules? Heyting stated on [He71, page 106]:

> It must be remembered that no formal system can be proved to represent adequately an intuitionistic theory. There always remains a residue of ambiguity in the interpretation of the signs, and it can never be proved with mathematical rigour that the system of axioms really embraces every valid method of proof.

Brouwer offered so-called weak counterexamples (Brouwerian counterexamples) to show that certain logical principles, like Excluded Middle $A \vee \neg A$, are not always intuitionistically valid. Bishop on [Bi67, page 9] introduced omniscience principles for the same purpose: If a certain logical statement A allows one to obtain a principle of omniscience, then that is accepted evidence that A cannot be proven constructively. The evidence for non-provability is obviously different in nature from provability. Although insufficient, one may read in the existence of Kripke model counterexamples to the intuitionistic provability of statements that such statements are plausibly not constructive. Consider a Kripke model $\mathfrak{A}$ with ordered collection of nodes $(W, \leq)$ and weakly increasing function $f : \mathrm{N} \to W$ of which we have no knowledge whether f ever increases or in what way. Function f simulates temporal information, where $f(m)$ indicates what we can know at stage m about our structure of interest. It is $\mathfrak{A}_\alpha$ when $f(m) = \alpha$, with the understanding that our knowledge may advance at a further stage $n > m$. At stage m we only can know what is forced at node $f(m)$. Insufficient, yes. Evidence of non-provability, also yes. If one accepts such models plus the Kripke model completeness theorem as evidence, then Heyting's set of axioms and rules is complete. A classical applied mathematician modeling an idealized intuitionist may certainly conclude this.

Intuitionistic predicate logic has been accepted by all major schools of constructivism that we listed. It is the internal logic of a topos. However, there is a *Third Question*: Are the axioms and rules of Heyting's intuitionistic predicate logic constructive?

4 Proof Interpretations for Intuitionistic Logic

Despite the common acceptance of intuitionistic predicate logic as the constructive one, there is no agreement on its justification.

Troelstra on [Tr81, page 16] cited a letter of Heyting to Oscar Becker dated 23 July 1933, in which Heyting wrote (with our bibliographical citation inserted)

> Ich habe die Axiome und Sätze der Principia mathematica [WR25] gesichtet und aus den zulässig befundenen ein System von unabhängigen Axiomen gemacht. Bei den relativen Vollständigkeit der Principia ist die Vollständigkeit meines Systems M.E. in der best möglichen Weise gesichert[3].

We follow [Tr81, page 16], where Troelstra stated that even if this seems a simple-minded procedure, it could "…be done by someone who had at least implicitly a clear grasp of the intuitionistic meaning of the logical operators." Let us expand the quote of Heyting at the beginning of Section 3: Heyting wrote on [He78, page 8] that

> Logic can be considered in different ways. As a study of regularities in language it is an experimental science which, like any such science, needs mathematical notions; therefore it belongs to applied mathematics. If we consider logic not from the linguistic point of view but turn our attention to the intended meaning, then logic expresses very general mathematical theorems about sets and their subsets.

Heyting addressed the intended meaning of the logical constants through his proof interpretation. This may answer the Third Question at the end of Section 3, and justify the axioms and rules of intuitionistic logic. Following [Tr81, page

[3] "I went through the axioms and theorems of principia mathematica, and made a system of independent axioms from the ones found acceptable. Because of the relative completeness of the one in principia is, in my opinion, the completeness of my system assured in the best possible way."

18], Heyting's clarification of the logical constants appeared as the result of scholarly debates which were already going on in 1928. His broadly recognized proof interpretation of the logical constants is in [He34]. The following version is from [TrvD88, page 9].

H1. A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.

H2. A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B (plus the stipulation that we want to regard the proof presented as evidence for $A \vee B$).

H3. A proof of $A \to B$ is a construction which permits us to transform any proof of A into a proof of B.

H4. Absurdity $\perp$ (contradiction) has no proof; a proof of $\neg A$ is a construction which transforms any hypothetical proof of A into a proof of a contradiction.

H5. A proof of $\forall x A(x)$ is a construction which transforms a proof of $d \in D$ (D the intended range of x) into a proof of $A(d)$.

H6. A proof of $\exists x A(x)$ is given by providing $d \in D$, and a proof of $A(d)$.

This is now known as the Brouwer-Heyting-Kolmogorov BHK interpretation. The rules of the interpretation are perceived as being used in the writings of Brouwer, the rules are due to Heyting, and Kolmogorov's so-called problem interpretation of [Ko32] for the proposition logical rules is often considered as substantially equivalent to Heyting's, see [He58]. The proof interpretation as stated is informal, and uses primitive terms like 'proof', 'construction', and 'hypothetical'.

On the one hand, Heyting's intuitionistic predicate logic has been accepted by the major schools of constructivism we listed before. Brouwer expressed appreciation for Heyting's intuitionistic logic, and supported the publications of [He30a, He30b, He30c], see [Tr81, page 17]. On the other hand, the proof interpretation has been challenged from multiple sides, mostly to refine or clarify Heyting's version, but not as a challenge[4] to Heyting's intuitionistic predicate logic.

Our original reason to deviate from the proof interpretation, and consequently from intuitionistic logic, comes from the use of 'construction' in the proof interpretation, both in the case H3 (and H4) for implication and in the case H5 for universal quantification. Our concerns aren't new. On [Tr77, page 977] we find a somewhat different proof interpretation. Its existence indicates the difficulty of finding a satisfactory justification for Heyting's predicate logic. The most significant differences with the proof interpretation above are the new required 'insights' in

H3'. A proof of $A \to B$ consists of a construction c which transforms any proof of A into a proof of B (*together* with the insight that c has the property: d proves A $\Rightarrow$ cd proves B).

H5'. ...we can explain a proof of $\forall x A x$ as a construction c which on application to any $d \in D$ yields a proof $c(d)$ of Ad, together with the insight that c has this property. ...

[4] See footnote 1 for some limited exceptions.

The origin of these 'insights' can be traced back[5] to Kreisel in [Kr62, Kr65], which Troelstra on [Tr81, page 20] described as

> Kreisel's attempts at a general theory of constructions and proofs.

On [Tr77, page 977] this variation on Heyting's proof interpretation is called the Brouwer-Heyting-Kreisel explanation. We appear to stay closer to Kreisel's original variation with the version (similar to [Ru91, page 156]):

H3". A proof p of $A \to B$ is a pair (q, r) such that q is a construction that converts proofs of A into proofs of B, and r is a proof[6] that q is such a construction.

H5". A proof p of $\forall x Ax$ is a pair (q, r) such that q is a construction which for each construction c of an element d of the domain D produces a proof $q(c)$ of Ad, and r is a proof that q is such a construction.

The clarification of implication is a key problem for constructivists. We illustrate this by examples from Markov constructivism, Bishop constructivism, and intuitionism. There is no such problem for constructivism along the lines of topos theory.

Kushner writes on [Ku06, page 565] about (Bishop and) Markov:

> ... [Bishop] could not avoid the key problem of any system of constructive mathematics, namely, the problem of clarifying implication. Markov spent the last years of his life struggling to develop a large "stepwise" semantic system in order to achieve, above all, a satisfactory theory of implication.

On [Bi67, page 7] Bishop wrote about the interpretation of implication (emphasis added):

> Statements formed with this connective, for example, statements of the type ((P implies Q) implies R), have a less immediate meaning than the statements from which they are formed, although in actual practice this does not *seem* to lead to difficulties in interpretation.

On [Bi70, page 56] Bishop wrote "The most urgent foundational problem of constructive mathematics concerns the numerical meaning of implication." Bishop continued on [Bi70, page 57] with

> ... I decided to let the mathematics be the test, and found that in actual practice there was little difficulty in giving numerical interpretations to statements with implications or even nested implications. Although the numerical meaning of implication is a priori unclear, in each particular instance the meaning is clear.

At its core the explanation of the meaning of implication is highly impredicative. As Michael Dummett wrote on [Du00, page 269] about the proof interpretation:

> The principal reason for suspecting these explanations of incoherence is their apparently highly impredicative character; if we know

[5] Kreisel's second clauses were an attempt to obtain decidability of the BHK clauses, and includes that a constructivist should recognize a constructive proof when (s)he sees one. The Kreisel-Goodman paradox (discovered by Kreisel and Goodman themselves by 1970) discredited the programme. However, see the revival of interest in [DeKu15].

[6] Called a Nachprüfungsbeweis on [Da82, page 60].

> which constructions are proofs of the atomic statements of any first-order theory, then the explanations of the logical constants, taken together, determine which constructions are proofs of any of the statements of that theory; yet the explanations require us, in determining whether or not a construction is a proof of a conditional or of a negation, to consider its effect when applied to an arbitrary proof of the antecedent or of the negated statement, so that we must, in some sense, be able to survey or grasp some totality of constructions which will include all possible proofs of a given statement.

For the foundations of constructive mathematics along the lines of topos theory, the proof interpretation appears of marginal importance. In this approach category theory is put forward as a foundation for mathematics. If intuitionistic logic as a formal system shows up as the internal logic of a topos, then that is a consequence of another foundational angle, a philosophy in which intuitionistic logic (and constructive mathematics) is a consequence rather than a source.

5 A General Theory of Constructions and Proofs

The collection of proofs has always been understood to be open-ended[7]. We demonstrate that open-ended implies that Basic logic rather than intuitionistic logic is the logic of constructive mathematics.

We introduce constructive logic anew by developing it simultaneously with a proof interpretation. Our method is axiomatic, so avoids unnecessary ontological commitments. The axioms constitute 'obvious' facts about constructions and proofs. We neither claim nor attempt that our axiomatics form a complete set for a full theory of constructions and proofs. We only claim that the resulting constructive logic may be considered complete.

We imagine an idealized constructivist who never makes mistakes, and has unlimited memory and 'time'. We build a system of rules and axioms which this constructivist accepts. We do not require the existence of a collection of all proofs. Instead, the accumulation of proofs is closed under certain canonical rules. We consider proof constructions mathematically. As Heyting wrote on [He31, page 114]:

> Ein Beweis für eine Aussage ist eine mathematische Konstruktion, welche selbst wieder mathematisch betrachtet werden kann[8].

Our theory of proofs is constructive itself, and is part of mathematics.

Kreisel in [Kr62, Kr65] developed a theory of constructions and proofs based on objects which one may describe as pairs (π, B) with π a proof object and B the proposition proved. Our objects consist of triples (A, π, B) where π is a proof of B from assumption A. The word 'assumption' replaces the use of 'hypothesis' of earlier proof interpretations so as to emphasize our distinct axiomatic approach. A constructivist may understand π as a construction which permits us to transform any proof of A into a proof of B. Triples (A, ξ, B)

[7] See the references farther down to Heyting on [He71, page 5], to van Atten on Brouwer on [vA18, page 3], and to Dummett on [Du00, page 274]. With some goodwill one may read on [Ku06, page 565] a recognition by Markov that the universe of proofs is open-ended. Goodman's stratification of proofs in [Gd70] (intended to avoid the Kreisel-Goodman paradox) may be another example.

[8] "A proof for a proposition is a mathematical construction, which itself again can be considered mathematically."

with ξ a proof variable are essential to our proof interpretation of formulas like $(A \to B) \to C$, where a proof of the principal implication needs an assumption of this triple form for $A \to B$ to express its meaning. We now believe that this insight isn't new. From the notes of his lecture at Zilsel's on [Go38, Go95, pages 100–101] we perceive that Gödel in 1938 suggested this ternary approach as alternative to Heyting's proof interpretation as well as to his own unary operator $\Box B$ of modal logic S4. We broaden the notation by allowing triples $(\vec{A}, \pi, B) = ((A_1, A_2, \ldots, A_n), \pi, B)$ with intended meaning that proof π permits assumptions from a list $\vec{A} = (A_1, A_2, \ldots, A_n)$. We allow $n = 0$. We may write $\vec{A}, A_{n+1}$ for $(A_1, A_2, \ldots, A_{n+1})$ or $A_0, \vec{A}$ for $(A_0, A_1, \ldots, A_n)$, and so on.

We axiomatize properties of ternary relation $(\vec{A}, \pi, B)$ with no attempt to decide the complete meaning of the words assumption or constructive. We defend our axioms individually as ones that should be accepted under sound constructive interpretations of these words. Proof π itself must display its assumptions and conclusion. Notation $(\vec{A}, \pi, B)$ is a clarification as to what these assumptions and conclusion are. For formal proofs we use symbols π, ρ, σ and so on, possibly with parameters. We write ξ, η, or ζ for proof variables. For formal statements we use A, B, C and so on, possibly with parameters. In our axiomatization below we occasionally ignore formula parentheses when it improves readability. We write $A \vdash_{\vec{B}} C$, with intended meaning C is derivable from $A, \vec{B}$, if we have a proof $(A, \vec{B}, \pi, C)$. We may write $A \vdash C$ when $A, \vec{B} = A, ()$, or $\vdash_{\vec{B}} C$ when $\vec{B}$ is the complete list. Below, each time when we introduce a new axiom about constructing new proofs from old, we add clarifications where necessary of the constructive meaning of 'assumption'. Our axioms about proofs respect constructive interpretation. We reject axioms that do not.

We make no attempt to have the axioms independent or the axiom system minimal. We start with a few axioms of our new proof interpretation which precede the ones for the usual logical constants. These axioms involve the constructive meaning of entailment $\vdash$. We freely reorder, or add and remove duplicate entries, in lists $\vec{A}$ of formulas. If we have a proof $(\vec{A}, \pi, C)$, then we also have a proof $(\vec{A}, B, \sigma, C)$. So, for example, we have rules

$$\frac{A \vdash_{\vec{D}} C}{A \vdash_{B,\vec{D}} C} \quad \text{and} \quad \frac{\vdash_{B,\vec{D}} C}{B \vdash_{\vec{D}} C} \quad \text{and} \quad \frac{B \vdash_{\vec{D}} C}{\vdash_{B,\vec{D}} C}$$

If a constructivist has assumption A, then A is accepted. This is a clarification of the intended meaning of 'assumption'. For each list of formulas $A, \vec{D}$ we have trivial proofs $(A, \vec{D}, \pi, A)$. So we have logical axiom schema[9]

$$A \vdash_{\vec{D}} A$$

Our new proof interpretation is axiomatic. Casper Storm Hansen on [Ha16, page 385] proposes an improved explanation of what Brouwer meant with truth

[9] On [vA18, page 4], in a remark referring to [Ru93], Mark van Atten claims that for Brouwerian intuitionists "... there is no such thing as a mere assumption. To assume that A is true is to assume that a construction for A has been carried out (perhaps in an idealised sense)". Mark van Atten points us to "Now suppose that $\vdash \neg p$, that is, we have deduced a contradiction from the supposition that p were carried out" on [He56, page 102] as his reference (personal communication). See also [He71, page 106] or [AtSu17]. The distinction is fictitious once the 'idealised sense' is understood sufficiently broadly to avoid a contradiction with Heyting's proof interpretation of negation. A proof of $\neg A$ is a construction that converts any hypothetical proof of A to a proof of $\perp$. When we have consistency, a hypothetical proof cannot become realized. As we understand it, this means that we assume A without any commitment to its truth. Otherwise we would never prove $\neg A$.

by distinguishing between what we may call proofs-in-content and proofs-as-anticipated. Our constructivist may interpret π as a construction which converts a construction for A into the same construction for A, be it an idealized construction for A or a proof-in-content or other.

We have a composition clause for proofs. Whenever a proposition B is proven using some assumptions, we are allowed to replace B as assumption in another derivation by its assumptions and proof. If (A, π, B) and (B, σ, C) are proofs, then so is $(A, \sigma \cdot \pi, C)$, also written as $(A, \sigma\pi, C)$, where $\sigma\pi$ stands for the composition proof, and which we construct in a uniform way in terms of π and σ. The constructive composition principle remains sound when the proofs π and σ employ additional assumptions: If we have proofs $(A, \vec{D}, \pi, B)$ and $(B, \vec{D}, \sigma, C)$, then also $(A, \vec{D}, \sigma\pi, C)$. So we have logical rule

$$\frac{A \vdash_{\vec{D}} B \quad B \vdash_{\vec{D}} C}{A \vdash_{\vec{D}} C}$$

5.1 Propositional Logic

Before considering full predicate logic, we restrict ourselves to a new proof interpretation for propositional logic. One advantage is that we can use simpler notation. We choose the usual logical constants $\wedge$, $\vee$, $\rightarrow$, $\top$, and $\perp$ of intuitionistic propositional logic. Negation $\neg A$ is defined by $A \rightarrow \perp$, and bi-implication $A \leftrightarrow B$ is defined by $(A \rightarrow B) \wedge (B \rightarrow A)$. It may be possible to add new ones, for example a new version of $\neg A$ distinct from $A \rightarrow \perp$. In this paper we consider no constructively distinct new proposition logical constants.

For each pair of formulas A and B we have a conjunction formula $A \wedge B$ with $\wedge$ the intended meaning of 'and'. There are trivial proofs $(A \wedge B, \pi_1, A)$ and $(A \wedge B, \pi_2, B)$. These come with the intended meaning of an assumption of form $A \wedge B$. Consequently, with composition, a proof $(C, \sigma, A \wedge B)$ implies that we have proofs $(C, \pi_1\sigma, A)$ and $(C, \pi_2\sigma, B)$. In the other direction, if we have proofs (C, π, A) and (C, σ, B), then there is a proof which we name $(C, \langle \pi, \sigma \rangle, A \wedge B)$, and which we construct in a uniform way in terms of π and σ. The constructive principles remain sound when the proofs employ additional assumptions. So we have rules

$$\frac{C \vdash_{\vec{D}} A \wedge B}{C \vdash_{\vec{D}} A} \quad \text{and} \quad \frac{C \vdash_{\vec{D}} A \wedge B}{C \vdash_{\vec{D}} B} \quad \text{and} \quad \frac{C \vdash_{\vec{D}} A \quad C \vdash_{\vec{D}} B}{C \vdash_{\vec{D}} A \wedge B}$$

By the intended meaning of $\wedge$, if a proof has assumptions A and B, then there is an essentially identical proof with assumption $A \wedge B$. We have a proof $(A, B, \vec{D}, \pi, C)$ exactly when we have a proof $(A \wedge B, \vec{D}, \sigma, C)$. So we have rules

$$\frac{A \vdash_{B, \vec{D}} C}{A \wedge B \vdash_{\vec{D}} C} \quad \text{and} \quad \frac{A \wedge B \vdash_{\vec{D}} C}{A \vdash_{B, \vec{D}} C}$$

We include a symbol $\top$ with intended meaning 'true', standing for a trivially proved statement. For every $\vec{D}$ there is a trivial proof $(\vec{D}, \pi, \top)$, and axiom

$$\vdash_{\vec{D}} \top$$

For each pair of formulas A and B we have a disjunction formula $A \vee B$ with $\vee$ the intended meaning of 'or'. There are trivial proofs $(A, \sigma_1, A \vee B)$ and $(B, \sigma_2, A \vee B)$. Consequently, with composition, a proof $(A \vee B, \pi, C)$ implies that we have proofs $(A, \pi\sigma_1, C)$ and $(B, \pi\sigma_2, C)$. In the other direction, if we have proofs (A, π, C) and (B, σ, C), then there is a proof which we name

$(A \vee B, [\pi, \sigma], C)$, and which we construct in a uniform way in terms of π and σ. This clarifies what it means to assume a disjunction $A \vee B$. The constructive principles remain sound when the proofs employ additional assumptions. So we have rules

$$\frac{A \vee B \vdash_{\vec{D}} C}{A \vdash_{\vec{D}} C} \quad \text{and} \quad \frac{A \vee B \vdash_{\vec{D}} C}{B \vdash_{\vec{D}} C} \quad \text{and} \quad \frac{A \vdash_{\vec{D}} C \quad B \vdash_{\vec{D}} C}{A \vee B \vdash_{\vec{D}} C}$$

Lists $\vec{D}$ of additional assumptions are key in proving distributivity. Straightforward proofs yield $B \vdash_A (A \wedge B) \vee (A \wedge C)$ and $C \vdash_A (A \wedge B) \vee (A \wedge C)$, so also

$$B \vee C \ \vdash_A \ (A \wedge B) \vee (A \wedge C)$$

Thus

$$A \wedge (B \vee C) \ \vdash_{\vec{D}} \ (A \wedge B) \vee (A \wedge C)$$

We include a symbol $\bot$ with intended meaning 'false'. Theories need not have an acceptable candidate for $\bot$. For significant theories like arithmetic an atom like $1 = 0$ can perform this role. In such cases the word 'false' is only used in an abstract sense. If we have such a candidate atom, then for every $\vec{D}$ and B we have a proof $(\bot, \vec{D}, \pi, B)$, and axiom

$$\bot \vdash_{\vec{D}} B$$

For each pair of formulas A and B we have an implication formula $A \rightarrow B$ with $\rightarrow$ the intended meaning of 'implies'. We write $\neg A$ as abbreviation for $A \rightarrow \bot$. Formula $A \rightarrow B$ has to reflect the meaning of $A \vdash B$ within the bounds of what is constructively acceptable. The critical issue of what these bounds are will be clarified below by permitted proof constructions on the one hand, and by weak counterexamples on the other. We needn't bother with a more complicated notation like $A \rightarrow_{\vec{D}} B$ for $A \vdash_{\vec{D}} B$, since the latter is equivalent to $(A \wedge \bigwedge \vec{D}) \rightarrow B$. If we have a proof $(A \wedge B, \pi, C)$, then we have a proof $(A, \pi_A, B \rightarrow C)$, where π_A takes proof π, replaces its assumption $A \wedge B$ by assumption B and derives $A \wedge B$ using the assumption A of π_A. Finally append that the result is a proof for conclusion $B \rightarrow C$. We construct π_A in a uniform way in terms of π. The constructive principle remains sound when the proof employs additional assumptions. So we have rule

$$\frac{A \wedge B \ \vdash_{\vec{D}} \ C}{A \ \vdash_{\vec{D}} \ B \rightarrow C}$$

Assume $A \rightarrow B$ and $B \rightarrow C$. So we assume proofs (A, ξ, B) and (B, η, C) without specifying ξ and η any further. As Bishop wrote on [Bi67, page 3]:

> Mathematics takes another leap, from the entity which is constructed in fact to the entity whose construction is hypothetical. To some extent hypothetical entities are present from the start: whenever we assert that every positive integer has a certain property, in essence we are considering a positive integer whose construction is hypothetical.

In this same sense ξ and η are hypothetical. From the assumed ξ and η we construct composition proof $(A, \eta\xi, C)$ in the hypothetical sense implied by Bishop. The constructive principle remains sound when the proof employs additional assumptions. So we have axiom

$$(A \to B) \wedge (B \to C) \vdash_{\vec{D}} A \to C$$

In special cases we may have proofs at hand. For example consider proof $(A \wedge B, \pi, B)$ instead of (A, ξ, B). In that case we get composition proof $(A \wedge B, \eta\pi, C)$, where proof $\eta\pi$ may only become available after mathematics advances and a proof σ can be substituted for η. This special case indicates that from $A \wedge B \vdash B$ we can derive $(B \to C) \vdash (A \wedge B \to C)$.

Assume $A \to B$ and $A \to C$. So we assume proofs (A, ξ, B) and (A, η, C). So we have proof $(A, \langle \xi, \eta \rangle, B \wedge C)$. The constructive principle remains sound when the proof employs additional assumptions, so we have axiom

$$(A \to B) \wedge (A \to C) \vdash_{\vec{D}} A \to (B \wedge C)$$

Assume $B \to A$ and $C \to A$. So we assume proofs (B, ξ, A) and (C, η, A). So we have proof $(B \vee C, [\xi, \eta], A)$. The constructive principle remains sound when the proof employs additional assumptions, so we have axiom

$$(B \to A) \wedge (C \to A) \vdash_{\vec{D}} (B \vee C) \to A$$

This completes our axiomatization when restricted to the language of propositional logic. Our system axiomatizes the Basic[10] Propositional Logic of Albert Visser in [Vi81], and is a proper subsystem of Intuitionistic Propositional Logic.

From a historical perspective, the constructive nature of Basic Propositional Logic is not controversial, only the matter of why we stop here is. We sketch below by means of general weak counterexamples why we stop here. To get Intuitionistic Propositional Logic it suffices to add schema $\top \to A \vdash A$ to Basic Propositional Logic. The schema allows one to derive modus ponens $A \wedge (A \to B) \vdash B$. Before discussing the general weak counterexamples, we show that our concerns are far from new, with a further focus on why $\top \to A \vdash A$ as a general principle should be excluded from constructive logic.

Heyting wrote on [He71, page 5] that

> ...one is never sure that the formal system represents fully any domain of mathematical thought; at any moment the discovering of new methods of reasoning may force us to extend the formal system.

This agrees with Brouwer's view. As Mark van Atten points out on [vA18, page 3], (with his citation adapted to our list of references)

> Intuitionists consider the notion of proof to be open-ended, not only epistemically (at no moment do we know all possible proofs) but ontologically, and hence they deny that there is such a thing as the totality of all intuitionistic proofs ([Br07, pages 148–149]). There is only a *growing* universe of mathematical objects and proofs.

Dummett wrote on [Du00, page 274]:

> As mathematics advances, we become able to conceive of new operations and to recognize them and others as effectively transforming proofs of B into proofs of C; and so the meaning of $B \to C$ would

[10] A slightly unfortunate name, and not unique in this. The related Basic Modal Logic (better known as K4) in the book [Sm85] of Craig Smoryński, refers to an essentially auxiliary modal logic for the purposes of its author. Another basic logic, of Giovanni Sambin, Giulia Battilotti, and Claudia Faggian in [SBF00], appears to have been named with purpose. For a unification of that one with our Basic Propositional Logic, see [ArVa12].

> change, if a grasp of it required us to circumscribe such operations in thought. Moreover, an operation which would transform any proof of $B \to C$ available to us now into a proof of D might not so transform proofs of $B \to C$ which became available to us with the advance of mathematics: and so what would now count as a valid proof of $(B \to C) \to D$ would no longer count as one.

We don't agree with Dummett's follow-up

> These fears are groundless. In order to recognize an operation as a proof of $(B \to C) \to D$, we must think of it as acting on anything we may ever recognize as a proof of $B \to C$. Of such a proof, we know in advance only what is specified by the intuitive explanation of $\to$: namely, that we recognize it as an effective operation, and as one that will transform any proof of B into a proof of C. We need not survey or circumscribe possible such operations in advance in any more particular way than this.

This is not an acceptable explanation. One doesn't know what to recognize as an effective operation as new methods of reasoning may be discovered. The following paragraph demonstrates that a survey or circumscription of possible future operations is unavoidable.

The principle $\top \to A \ \vdash \ A$ as a general mathematical theorem fails to respect the observations of Heyting, van Atten, and Dummett when applied to a general theory of constructions and proofs. For comparison consider the case $(A \to B) \wedge (B \to C) \vdash (A \to C)$, which we justify by converting assumptions (A, ξ, B) and (B, η, C) into $(A, \eta\xi, C)$. Composition $\eta\xi$ exists without a need to 'look inside' ξ and η, so is constructively justified. All Basic logic axioms and rules are elementary, or are justified by building new constructions from old ones without a need to 'look inside' them. By contrast, consider a proof $((\top \to A), \pi, A)$. This is different from (A, ρ, A), where we may assume a proof of A in an idealized sense[11], we always conclude A from its assumption. Construction π claims to turn an assumed proof $(\top, \xi, A)$ of A into a proof of A. Suppose that a construction σ appears that fulfills $(\top, \sigma, A)$. A constructivist understands σ as a construction which permits us to form a proof of A. This may be understood before performing σ. A constructivist doesn't recognize a proof unseen. In the absence of other reasons why A has a proof, we must perform σ to form a proof of A. This could be harmless were it not that the accumulation of proofs is open-ended. After sufficient completion of σ we may discover new methods of reasoning. Such new methods of reasoning need not be part of performing π when transforming a different $(\top, \tau, A)$ into a proof of A. The existence of π is claimed prematurely. When we assume not a proof of A but rather assume that we have a construction for a proof of A, then we have a construction for a proof of A, which gives us $\top \to A \ \vdash \ \top \to A$.

Despite the absence of full modus ponens, limited versions still hold. For example $\vdash \ A \to B$ implies $A \vdash B$, see [Vi81]. So Basic Propositional Logic satisfies the rule

$$\frac{C \vdash_{\vec{D}} A \qquad \vdash A \to B}{C \vdash_{\vec{D}} B}$$

As a special case, $\vdash \top \to A$ implies $\vdash A$. From [ArRu98, page 323] we see that Basic Propositional Logic satisfies $\top \to A \ \vdash \ A$ exactly when A is equivalent to a formula of the form $((\top \to B) \to B) \to (\top \to B)$.

[11] For example by supposing that a proof of A has been carried out. See also footnote 9.

Do we have a complete set of axioms and rules for constructive propositional logic? As in the intuitionistic case, the evidence for non-provability is different in nature from provability. Both in [Vi81] and in [ArRu98] we find a completeness theorem for Basic Propositional Logic with transitive Kripke models, that is, Kripke models where the world relation is transitive but not necessarily reflexive as in the case of Intuitionistic Propositional Logic. Using transitive Kripke models as weak counterexamples to constructive provability of propositional statements has similar limited value as when using reflexive transitive Kripke models as a tool to make weak counterexamples in the intuitionistic case. Imagine a Kripke model $\mathfrak{A}$ on a structure $(W, \sqsubset)$ of nodes or worlds W, with transitive relation $\sqsubset$, and with structures $\mathfrak{A}_\alpha$ at nodes $\alpha \in W$. A weakly increasing function $f : \mathrm{N} \to W$ simulates temporal information, where what is forced at $f(n) = \alpha$ indicates what the constructivist can know at stage n about a structure of interest. The constructivist has no knowledge whether f ever increases or in what way. Each node α represents what a constructivist can know with the methods of reasoning at that node, and with the knowledge at that node about the intended structure. At node α, nodes $\beta \sqsupset \alpha$ represent possible versions of knowledge with newly conceived operations or further understanding about the intended structure. In terms of the proof interpretation, at node β constructions c at stage α for implications $A \to B$ have been sufficiently completed to obtain proofs (A, π, B). The superstructure of such nodes and structures above α present the limits on what a constructivist can know at node α. Kripke models as so described may be seen as classical 'platonistic' (in the sense that possible future stages are all held in common existence) models of evidence which may convince constructivists that certain logical statements are not derivable. By definition we have $\alpha \Vdash A \to B$ exactly when for all $\beta \sqsupset \alpha$ we have $\beta \Vdash A$ implies $\beta \Vdash B$. By definition we have $\alpha \Vdash (A \vdash B)$ exactly when for all $\beta \sqsupseteq \alpha$ we have $\beta \Vdash A$ implies $\beta \Vdash B$. Situation $\alpha \not\sqsubset \alpha$ of an irreflexive node simulates a constructivist who is in possession of constructions, maybe idealized or in anticipation, of proofs which may lead to newly conceived operations or to further understanding about the intended structure. If we further have $\beta \Vdash A$ for all $\beta \sqsupset \alpha$, we simulate that at node α the constructivist has a construction, maybe idealized or in anticipation, of a proof of A. So $\alpha \Vdash \top \to A$. Subcase $\alpha \Vdash A$ simulates that the constructivist has a proof (construction) for A. Subcase $\alpha \nVdash A$ simulates that the constructivist has a construction c for a proof (construction) of A which, when c has been sufficiently completed, reveals a proof (construction) for A which employs new methods of reasoning. An irreflexive node α with $\alpha \Vdash A \to B$ allows us to simulate a constructivist who can have a construction c, in anticipation or in content or other, for a proof (A, ξ, B) such that after sufficient completion of c a proof (A, π, B) is produced with possibly new methods of reasoning. Such π are again constructions. Consider the special case where $\alpha \Vdash A \to (B_1 \to B_2)$. In that case sufficient completion of c results in a proof $(A, \pi, B_1 \to B_2)$. If $\alpha \sqsubset \beta \Vdash A$, then $\beta \Vdash B_1 \to B_2$, which simulates that at this further stage there is a construction d for (B_1, η, B_2). For irreflexive β this allows for the possibility that only after further completion of d we get a proof (B_1, σ, B_2) employing still further new methods of reasoning. Are transitive Kripke models insufficient? Yes. Are they evidence of non-provability? Also yes. If one is willing to accept the transitive Kripke models and the completeness theorem as evidence, then Basic Propositional Logic is complete.

Over Basic Propositional Logic we have the equivalence of $\neg\neg A$ with $\neg\neg\neg\neg A$, see [AAR16, Proposition 4.1.4, page 145]. Brouwer writes in [Br24] that even

$\neg A$ and $\neg\neg\neg A$ are equivalent. The key step[12] in his argument is that A implies $\neg\neg A$. We follow Heyting and equate $\neg A$ with $A \to \bot$. A special case of the key step is the claim $\top \ \vdash \ (\top \to \bot) \to \bot$ which over Basic Propositional Logic is equivalent to $(\top \to \bot) \ \vdash \ \bot$ since Basic Propositional Logic is faithful, see [ArRu98, pages 321 and 329]. Using our 'axiomatic' notation, the key step implies that there exists a constructive proof $((\top \to \bot), \pi, \bot)$ which turns a hypothetical proof of inconsistency into an actual proof of a contradiction. This is too broad an acceptance of proof-in-principle for a growing universe of mathematical objects and proofs since such a future (hypothetical) construction may not be constructively acceptable at present.

5.2 Predicate Logic

When we broaden from propositional logic to predicate logic, we extend from propositional letters P to predicates $P(x_1, x_2, \ldots, x_m)$ of arities $m \geq 0$. For convenience we may write $\mathbf{x}$ for lists $x_1, x_2, \ldots, x_m$ of variables of finite length $m \geq 0$. We write $\mathbf{xy}$ for concatenated lists $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$. We write $P(\mathbf{x})$ for atoms $P(x_1, x_2, \ldots, x_m)$, and $A\mathbf{x}$ for formulas A when we want to emphasize that all free variables of A are among those in $\mathbf{x}$. For constant symbols $c_1, c_2, \ldots, c_k$ we use similar conventions.

Constant symbols are intended to stand for descriptions. The word 'description' replaces the use of 'construction' or other terminology of earlier proof interpretations in describing elements of the domain of discourse. On [Bi67, pages 6–7] Bishop uses the word 'description' for sets defined in a possibly incomplete way. We broaden this approach to all objects. Logic expresses very general mathematical theorems which includes theorems about mathematics where descriptions may be less complete. Descriptions may be idealized constructions, or ones in anticipation. The universe of descriptions itself may be an open-ended accumulation like the universe of proofs, we may discover new descriptions as mathematics advances.

Variables x are intended to range over descriptions including ones that are not yet recognized to be of elements of the intended domain. We write $\mathrm{E}\,x$ or $\mathrm{E}(x)$ for the propositional statement that the element described by x is sound, that is, the element described by x belongs to the intended domain. This is in line with Dana Scott's existence predicate of [Sc79]. We may write $\mathrm{E}\,\mathbf{x}$ or $\mathrm{E}(\mathbf{x})$ for $\mathrm{E}\,x_1 \wedge \mathrm{E}\,x_2 \wedge \ldots \wedge \mathrm{E}\,x_n$. We treat existence as a primitive $\mathrm{E}\,x$, thereby preserving optimal generality when axiomatizing our very general mathematical theorems.

We write $(\vec{A}, \pi\mathbf{x}, B)$ for a proof $\pi\mathbf{x}$ with assumptions $\vec{A}$ and conclusion B, where list $\mathbf{x}$ includes all variables that occur free in $\vec{A}, B$. Besides assumptions $\vec{A}$ and conclusion B, proof $\pi\mathbf{x}$ also holds all substitution places of variables $\mathbf{x}$ in those formulas. We write $A \vdash_{\mathbf{x},\vec{B}} C$ if a proof $(A, \vec{B}, \pi\mathbf{x}, C)$ exists, with the expected abbreviations when A is absent or $\vec{B} = ()$ or $\mathbf{x} = ()$. For convenience we ignore the order of the variables in $\mathbf{x}$, or duplications among them. Proof $\pi\mathbf{x}$ is such that no further meaning is assigned to the variables x_i in $\mathbf{x}$ in deriving C, beyond what follows from assuming $A, \vec{B}$. This clarifies what it means to assume formulas with free variables.

We allow the 'empty' description ℓ, a description of 'nothing', an extreme version of an incomplete description for elements of the intended domain. So there is at least one description. There is no element in the intended domain

[12] The constructive validity of this step has been challenged before, see [Go33, Go95, page 53].

described by ℓ. One technical advantage of ℓ is that variables range over at least one object, see the relevant substitution schema below where a free variable y is replaced by ℓ. The atomic existence sentence $\mathrm{E}\,\ell$ can play the role of an abstract symbol $\bot$. We have $\bot$ in the language, and a trivial proof $(\mathrm{E}\,\ell, \pi, \bot)$. So also axiom

$$\mathrm{E}\,\ell \;\vdash_{\mathbf{x},\vec{D}}\; \bot$$

If y isn't specified in proof $\pi\mathbf{x}$, then we can modify the proof to $\pi'\mathbf{x}y$ by adding such y. If, instead, variable y in proof $(A, \vec{D}, \pi\mathbf{x}y, C)$ doesn't occur in $A, \vec{D}, C$, then we can remove y, for example by replacing y by the empty description ℓ. So we have rules

$$\frac{A \;\vdash_{\mathbf{x}y,\vec{D}}\; C}{A \;\vdash_{\mathbf{x},\vec{D}}\; C} \quad y \text{ not free in } A, \vec{D}, C, \quad \text{and} \quad \frac{A \;\vdash_{\mathbf{x},\vec{D}}\; C}{A \;\vdash_{\mathbf{x}y,\vec{D}}\; C}$$

No further meaning is assigned to the variables x_i in $\mathbf{x}$ in deriving C, beyond what follows from assuming $A, \vec{D}$, so we can substitute other free variables for them. A proof $(A\mathbf{x}, \vec{D}\mathbf{x}, \pi\mathbf{x}, C\mathbf{x})$ implies a proof $(A\mathbf{y}, \vec{D}\mathbf{y}, \pi\mathbf{y}, C\mathbf{y})$, where no variable of $\mathbf{y}$ is allowed to become bound after substitution. So we have substitution rule

$$\frac{A\mathbf{x} \;\vdash_{\mathbf{x},\vec{D}\mathbf{x}}\; C\mathbf{x}}{A\mathbf{y} \;\vdash_{\mathbf{y},\vec{D}\mathbf{y}}\; C\mathbf{y}} \quad \text{no variable of } \mathbf{y} \text{ becomes bound}$$

Substitution is more general than renaming free variables since we may substitute the same variable y for different variables x and x'. Substitution by constants is easier. So we have substitution rule

$$\frac{A\mathbf{xy} \;\vdash_{\mathbf{xy},\vec{D}\mathbf{xy}}\; C\mathbf{xy}}{A\mathbf{xc} \;\vdash_{\mathbf{x},\vec{D}\mathbf{xc}}\; C\mathbf{xc}}$$

We have the expected generalizations of the rules preceding Subsection 5.1, essentially by adding variables $\mathbf{x}$ where necessary or permitted. For example, we have rules

$$\frac{A \vdash_{\mathbf{x}.\vec{D}} C}{A \vdash_{\mathbf{x},B,\vec{D}} C} \quad \text{and} \quad \frac{\vdash_{\mathbf{x},B,\vec{D}} C}{B \vdash_{\mathbf{x},\vec{D}} C} \quad \text{and} \quad \frac{B \vdash_{\mathbf{x},\vec{D}} C}{\vdash_{\mathbf{x},B,\vec{D}} C}$$

where $\mathbf{x}$ includes all free variables that occur. Similarly for axiom and rule

$$A \vdash_{\mathbf{x},\vec{D}} A \quad \text{and} \quad \frac{A \vdash_{\mathbf{x},\vec{D}} B \quad B \vdash_{\mathbf{x},\vec{D}} C}{A \vdash_{\mathbf{x},\vec{D}} C}$$

The proposition logical rules of derivations for $A \wedge B$, $\top$, $A \vee B$, and $\bot$ also remain essentially unchanged. In principle the same holds for implication $A \to B$. However, we replace the proposition logical rules for implication by different ones when we combine implication with universal quantification.

Predicates P stand for statements about elements of the intended domain. So predicates are strict, that is, we have abstract axiom schemas of form $(P(\mathbf{x}), \pi\mathbf{x}, \mathrm{E}\,\mathbf{x})$ and axioms

$$P(\mathbf{x}) \;\vdash_{\mathbf{xy},\vec{D}}\; \mathrm{E}\,\mathbf{x}$$

A proof interpretation presupposes the constructive nature of atomic statements. So for the axiomatized object $\pi\mathbf{x}$ evidence must be substituted in concrete situations.

For each formula A we have formulas $\exists xA$ with $\exists x$ the intended meaning 'there exists x'. We freely rename variables bound by $\exists$, with the usual restrictions that with substitution the new variables don't become bound by other quantifiers, and unchanged variables don't become bound by the new variable attached to this $\exists$. There are trivial proofs $(A \wedge \mathrm{E}\, x, \sigma x\mathbf{y}, \exists xA)$. Suppose we have a proof $(\exists xA, \pi\mathbf{y}, B)$, where x doesn't occur in $\mathbf{y}$. We can trivially modify the proof by adding a variable so that we have $(\exists xA, \pi x\mathbf{y}, B)$. By composition we have a proof $(A \wedge \mathrm{E}\, x, (\pi\sigma)x\mathbf{y}, B)$. In the other direction, suppose we have a proof $(A \wedge \mathrm{E}\, x, \pi x\mathbf{y}, B)$, where x isn't free in B. Then there is a proof which we name $(\exists xA, [\pi]\mathbf{y}, B)$, and which we construct in a uniform way in terms of $\pi x\mathbf{y}$. This clarifies what it means to assume $\exists xA$. Further assumptions $\vec{D}$ are permitted when they don't limit the meaning of x. So we have rules

$$\frac{A \wedge \mathrm{E}\, x \ \vdash_{x\mathbf{y},\vec{D}} \ B}{\exists xA \ \vdash_{\mathbf{y},\vec{D}} \ B} \quad x \text{ not free in } B, \vec{D} \quad \text{and} \quad \frac{\exists xA \ \vdash_{\mathbf{y},\vec{D}} \ B}{A \wedge \mathrm{E}\, x \ \vdash_{x\mathbf{y},\vec{D}} \ B}$$

Lists $\vec{D}$ of additional assumptions are key in proving existential distributivity. A straightforward proof yields $B \wedge \mathrm{E}\, x \ \vdash_{x\mathbf{y},A} \ \exists x(A \wedge B)$. So if x is not free in A, then

$$\exists xB \ \vdash_{\mathbf{y},A} \ \exists x(A \wedge B)$$

Thus

$$A \wedge \exists xB \ \vdash_{\mathbf{y},\vec{D}} \ \exists x(A \wedge B) \qquad x \text{ not free in } A$$

We introduce implication and universal quantification using one single format. The combination formulas are essentially more elaborate than the form $\forall xA$. Implications $A \to B$ are definable in a uniform way as special cases.

We briefly employ proposition logical implication to clarify the reasons for the choice of a new format. Suppose we have a proof $(A \wedge \mathrm{E}\, x, \pi x\mathbf{y}, B)$ with x not free in A. So we have a proof $(A, \pi_A x\mathbf{y}, \mathrm{E}\, x \to B)$, where $\pi_A x\mathbf{y}$ takes proof $\pi x\mathbf{y}$, replaces its assumptions $A \wedge \mathrm{E}\, x$ by assumption $\mathrm{E}\, x$ and derives $A \wedge \mathrm{E}\, x$ using the assumption A of $\pi_A x\mathbf{y}$. Finally append that the result is a proof of conclusion $\mathrm{E}\, x \to B$. Since variable x isn't free in A, construction $\pi_A x\mathbf{y}$ works for any description c one may ever substitute for x, where $\mathrm{E}\, c$ restricts the choice of descriptions to those of elements assumed to belong to the intended domain. So we have a proof $(A, \sigma\mathbf{y}, \forall x(\mathrm{E}\, x \to B))$ with $\forall x$ the intended meaning 'for all x'. Basic logic lacks full modus ponens, so the arrow in $\mathrm{E}\, x \to B$ is no longer always removable over domains with elements. This differs from the case of existential quantification, where formulas $\exists xA$ and $\exists x(\mathrm{E}\, x \wedge A)$ remain equivalent. Quantification is intended to only apply to descriptions of elements of the intended domain, so formula $\forall x(\mathrm{E}\, x \to B)$ is equivalent to $\forall x(\top \to B)$. We employ two ways to broaden our new notation. First, by the same argument why it is constructively acceptable to conclude $A \vdash_{\mathbf{y},\vec{D}} \forall x(\top \to C)$ from $A \wedge \mathrm{E}\, x \vdash_{x\mathbf{y},\vec{D}} C$ (x not free in $A, \vec{D}$), it is constructively acceptable to conclude $A \vdash_{\mathbf{y},\vec{D}} \forall x(B \to C)$ from $A \wedge B \wedge \mathrm{E}\, x \vdash_{x\mathbf{y},\vec{D}} C$ (x not free in $A, \vec{D}$). Second, since nested quantifications like $\forall x \forall y(A \to B)$ aren't well defined, we include multi-variable universal quantifications like $\forall xy(A \to B)$ as primitive new notations.

Combining the motivations of the previous paragraph, we have for each pair of formulas A and B universal implication formulas $\forall\mathbf{x}(A \to B)$ with the intended meaning that for all lists $\mathbf{x}$ assumed in the intended domain and for which A is assumed to hold, formula B also holds. Formula $\forall\mathbf{x}(A \to B)$

has to reflect the meaning of $A \wedge \mathrm{E}\,\mathbf{x} \vdash_{\mathbf{xy},\vec{D}} B$ within the bounds of what is constructively acceptable. We freely rename variables bound by $\forall$, with the restrictions that with substitution the new variables don't become bound by other quantifiers or by each other, and unchanged variables don't become bound by the new variables attached to this $\forall$. For convenience we ignore the order of the variables in $\mathbf{x}$, or duplications among them, in our notation. List $\mathbf{x}$ is allowed to have length 0. In that case we write $A \to B$ as short for $\forall(A \to B)$.

Suppose we have a proof $(A \wedge B \wedge \mathrm{E}\,\mathbf{x}, \pi\mathbf{xy}, C)$ with none of the variables in list $\mathbf{x}$ free in A. Then we have a proof $(A, \pi_A\mathbf{xy}, (B \wedge \mathrm{E}\,\mathbf{x}) \to C)$, where $\pi_A\mathbf{xy}$ takes proof $\pi\mathbf{xy}$, replaces its assumption $A \wedge B \wedge \mathrm{E}\,\mathbf{x}$ by assumption $B \wedge \mathrm{E}\,\mathbf{x}$ and derives $A \wedge B \wedge \mathrm{E}\,\mathbf{x}$ using the assumption A of $\pi_A\mathbf{xy}$. Finally append that the result is a proof for conclusion $(B \wedge \mathrm{E}\,\mathbf{x}) \to C$. Since variables $\mathbf{x}$ aren't free in A, construction $\pi_A\mathbf{xy}$ works for any list of descriptions $\mathbf{c}$ one may ever substitute for the $\mathbf{x}$, where $\mathrm{E}\,\mathbf{c}$ restricts the choice of descriptions for B and C to those of elements assumed to belong to the intended domain. So we have a proof which we name $(A, \langle\pi_A\rangle\mathbf{y}, \forall\mathbf{x}(B \to C))$, and which we construct in a uniform way in terms of $\pi_A\mathbf{xy}$. Further assumptions $\vec{D}$ are permitted when they don't limit the meaning of $\mathbf{x}$. Thus we have rule

$$\frac{A \wedge B \wedge \mathrm{E}\,\mathbf{x} \vdash_{\mathbf{xy},\vec{D}} C}{A \vdash_{\mathbf{y},\vec{D}} \forall\mathbf{x}(B \to C)} \quad \text{no variable of } \mathbf{x} \text{ free in } A, \vec{D}$$

In particular $A \vdash_{\mathbf{y},\vec{D}} B \to C$ follows from $A \wedge B \vdash_{\mathbf{y},\vec{D}} C$.

Universal implication broadens the rules for proposition logical implication. We get natural replacements for its proposition logical 'formalization' axioms. The earlier axioms for implication are replaced by

$$\begin{aligned}
&\forall\mathbf{x}(A \to B) \wedge \forall\mathbf{x}(B \to C) \vdash_{\mathbf{y},\vec{D}} \forall\mathbf{x}(A \to C)\\
&\forall\mathbf{x}(A \to B) \wedge \forall\mathbf{x}(A \to C) \vdash_{\mathbf{y},\vec{D}} \forall\mathbf{x}(A \to (B \wedge C))\\
&\forall\mathbf{x}(B \to A) \wedge \forall\mathbf{x}(C \to A) \vdash_{\mathbf{y},\vec{D}} \forall\mathbf{x}((B \vee C) \to A)
\end{aligned}$$

Although their justifications are similar to the ones for implication in Subsection 5.1, there are complications which we could avoid in the proposition logical case. Consider a formula $\forall\mathbf{x}(A\mathbf{xy} \to B\mathbf{xy})$, where $\mathbf{xy}$ includes all variables free in A or B, and lists $\mathbf{x}$ and $\mathbf{y}$ have no variables in common. When we assume this formula, the variables in $\mathbf{x}$ and $\mathbf{y}$ have different status. We have an assumed proof $(A\mathbf{xy} \wedge \mathrm{E}\,\mathbf{x}, (\xi\mathbf{x})\mathbf{y}, B\mathbf{xy})$ with ξ a 'hypothetical' proof variable, and the parentheses in notation $(\xi\mathbf{x})$ indicate that $\mathbf{x}$ lists the variables over which ξ is assumed to universally quantify. We assume a construction γ of a proof (construction) $(\xi\mathbf{x})\mathbf{y}$. Construction γ may be idealized or in anticipation. Its intended meaning is as follows. We describe two possible scenarios, one focused on variables $\mathbf{x}$, the other focused on variables $\mathbf{y}$. Consider the variables $\mathbf{x}$. After sufficient completion of γ we may obtain a proof for ξ such that from now on, but not before, we have that for all descriptions $\mathbf{c}$ of elements of the domain, we can perform composition $\xi\mathbf{c}$ and have $(\xi\mathbf{c})\mathbf{y}$. If $\mathbf{y} = ()$ is empty, then $(\xi\mathbf{c})$ is a proof of $A\mathbf{c}() \vdash B\mathbf{c}()$. Now consider the variables $\mathbf{y}$ instead. For all descriptions $\mathbf{d}$ we have a proof $(\xi\mathbf{x})\mathbf{d}$, still under construction, of $A\mathbf{xd} \vdash B\mathbf{xd}$. Without sufficient completion of γ, proof $(\xi\mathbf{x})\mathbf{d}$ may still be 'hypothetical'. Even if $\mathbf{x} = ()$ is empty and descriptions $\mathbf{d}$ become available after some advancement in mathematics, we may at most have assumption $A()\mathbf{d} \to B()\mathbf{d}$. Only after a sufficient completion of γ we may obtain a proof for $(\xi)\mathbf{d}$ of $A()\mathbf{d} \vdash B()\mathbf{d}$. This clarifies what it means to assume $\forall\mathbf{x}(A\mathbf{xy} \to B\mathbf{xy})$. This is essentially more general than a proof for $A\mathbf{xy} \wedge \mathrm{E}\,\mathbf{x} \vdash B\mathbf{xy}$, from which we obtain a proof for $A\mathbf{xy} \wedge \mathrm{E}\,\mathbf{xy} \vdash B\mathbf{xy}$

where variables $\mathbf{x}$ and $\mathbf{y}$ have the same status. For the first axiom above of 'formalized' composition consider assumed proofs $(A \wedge \mathrm{E}\,\mathbf{x}, (\xi\mathbf{x})\mathbf{y}, B)$ and $(B \wedge \mathrm{E}\,\mathbf{x}, (\eta\mathbf{x})\mathbf{y}, C)$, where we may suppose that $\mathbf{x}$ and $\mathbf{y}$ are disjoint. The composition proof equals $(A \wedge \mathrm{E}\,\mathbf{x}, ((\eta\xi)\mathbf{x})\mathbf{y}, C)$ with the distinct roles of the variables in $\mathbf{x}$ and $\mathbf{y}$ preserved. The same applies to the other two axioms.

When we broaden from propositional logic to predicate logic, further 'formalization' axioms are needed.

Assume $\forall\mathbf{x}(A \to B)$. So we assume we have a proof $(A \wedge \mathrm{E}\,\mathbf{x}, (\xi\mathbf{x})\mathbf{z}, B)$ with ξ as proof variable. We can substitute new variables $\mathbf{y}$ for the ones in $\mathbf{x}$ with the usual restrictions that none of them become bound in A or B, and that $\mathbf{y}$ and $\mathbf{z}$ are disjoint. So we have a proof $(A[\mathbf{x}/\mathbf{y}] \wedge \mathrm{E}\,\mathbf{y}, (\xi\mathbf{y})\mathbf{z}, B[\mathbf{x}/\mathbf{y}])$. Thus we have axiom

$$\forall\mathbf{x}(A\mathbf{x} \to B\mathbf{x}) \;\vdash_{\mathbf{z},\vec{D}}\; \forall\mathbf{y}(A\mathbf{y} \to B\mathbf{y}) \quad \text{no variable in } \mathbf{y} \text{ becomes bound in } A \text{ or } B\text{, and } \mathbf{y} \text{ and } \mathbf{z} \text{ are disjoint}$$

This axiom is more general than renaming variables. For example we also have

$$\forall xx'(Axx' \to Bxx') \;\vdash\; \forall y(Ayy \to Byy)$$

subject to the usual variable substitution restrictions.

For the existential quantifier $\exists$ we need a further such 'formalization' axiom. Assume $\forall\mathbf{x}y(A \to B)$, with y not free in B. So we assume we have a proof $(A \wedge \mathrm{E}\,\mathbf{x}y, (\xi y\mathbf{x})\mathbf{z}, B)$ with ξ as 'hypothetical' proof variable. So we have a proof $(\exists yA \wedge \mathrm{E}\,\mathbf{x}, ([\xi]\mathbf{x})\mathbf{z}, B)$. Thus we have axiom

$$\forall\mathbf{x}y(A \to B) \;\vdash_{\mathbf{z},\vec{D}}\; \forall\mathbf{x}(\exists yA \to B) \quad y \text{ not free in } B$$

Assuming a universal implication $\forall\mathbf{x}(A \to B)$ implies that we have an assumed proof $(A \wedge \mathrm{E}\,\mathbf{x}y, (\xi\mathbf{x}y)\mathbf{z}, B)$, where we choose to add a variable y not free in A or B. Absence of a free y in A or B allows this assumption of trivial universal quantification of ξ over y. So we have axiom

$$\forall\mathbf{x}(A \to B) \;\vdash_{\mathbf{z},\vec{D}}\; \forall\mathbf{x}y(A \to B) \quad y \text{ not free in any of the formulas}$$

There are trivial proofs $(\forall\mathbf{x}y(A \to B), \pi y\mathbf{z}, \forall\mathbf{x}((A \wedge \mathrm{E}\,y) \to B))$. These come with the intended meaning of assuming a universal implication $\forall\mathbf{x}y(A \to B)$, in this case an assumed proof $(A \wedge \mathrm{E}\,\mathbf{x}y, (\xi\mathbf{x}y)\mathbf{z}, B)$. Replace $(\xi\mathbf{x}y)\mathbf{z}$ by $(\eta\mathbf{x})y\mathbf{z}$ by ignoring the universal quantification over y. Consequently, with composition, a proof $(C, \sigma y\mathbf{z}, \forall\mathbf{x}y(A \to B))$ implies that we have a proof

$$(C, (\pi\sigma)y\mathbf{z}, \forall\mathbf{x}((A \wedge \mathrm{E}\,y) \to B))$$

Further assumptions $\vec{D}$ are permitted. So we have rule

$$\frac{C \;\vdash_{y\mathbf{z},\vec{D}}\; \forall\mathbf{x}y(A \to B)}{C \;\vdash_{y\mathbf{z},\vec{D}}\; \forall\mathbf{x}((A \wedge \mathrm{E}\,y) \to B)}$$

This completes our axiomatization of Basic Predicate Logic.

The system includes the Basic Predicate Calculus BQC of [Ru98] when we ignore function symbols and equality. Over our new system we have both $\exists xA \;\vdash\; \exists x(\mathrm{E}\,x \wedge A)$ and $\exists x(\mathrm{E}\,x \wedge A) \;\vdash\; \exists xA$, and both $\forall\mathbf{x}(A \to B) \;\vdash\; \forall\mathbf{x}((A \wedge \mathrm{E}\,\mathbf{x}) \to B)$ and $\forall\mathbf{x}((A \wedge \mathrm{E}\,\mathbf{x}) \to B) \;\vdash\; \forall\mathbf{x}(A \to B)$. So BQC without existence predicate but with inhabited intended domain embeds in the new system by relativization of quantifiers plus axiom $\vdash \exists x\top$. Additionally replace entailments $A \vdash B$ of BQC by $A \wedge \mathrm{E}\,\mathbf{x} \vdash B$, where $\mathbf{x}$ lists the free variables of A and B.

Conversely, embed the new system in BQC by adding a new predicate Fx to the language of BQC, and then relativize quantifiers to F while sending E to F. The presence of ℓ makes that over the BQC language with extra predicate Fx the overall domain has an element, as is required.

Do we have a complete set of axioms and rules for constructive predicate logic? The evidence for non-provability is different in nature from provability. In [Ru98] we find a completeness theorem for BQC with transitive Kripke models. There is no Kripke model completeness theorem for the new system. Using transitive Kripke models as weak counterexamples to constructive provability has limited value as do reflexive transitive Kripke models in the intuitionistic case. Imagine a Kripke model $\mathfrak{M}$ on $(W, \sqsubset)$ with nodes W and transitive relation $\sqsubset$, and with structures $\mathfrak{M}_\alpha$ at nodes $\alpha \in W$ with 'extended' domains M_α. The intended proper sub-domain at α is determined by those c for which $\alpha \Vdash \mathrm{E}\,c$. All relations from the predicate logic language are essentially restricted to these sub-domains. For all $\alpha \sqsubseteq \beta$ we have morphisms φ^α_β such that $\varphi^\alpha_\alpha = \mathrm{id}$, and $\varphi^\alpha_\gamma = \varphi^\beta_\gamma \varphi^\alpha_\beta$ for all $\gamma \sqsupseteq \beta \sqsupseteq \alpha$. A weakly increasing function $f : \mathrm{N} \to W$ simulates temporal information, where what is forced at $f(n) = \alpha$ indicates what the constructivist can know at stage n. Parts of this are motivated in the proposition logical case on page 15. We have $\alpha \Vdash_{\mathbf{y}} (A\mathbf{y} \vdash_{\mathbf{y}} B\mathbf{y})$ exactly when for all $\beta \sqsupseteq \alpha$ and $\mathbf{c} \in M_\beta$ we have $\beta \Vdash A\mathbf{c}$ implies $\beta \Vdash B\mathbf{c}$. So we have $\alpha \Vdash_{\mathbf{y}} \forall\mathbf{x}(A\mathbf{xy} \to B\mathbf{xy})$ exactly when for all $\beta \sqsupseteq \alpha$ and $\mathbf{c} \in M_\beta$ we have $\beta \Vdash \forall\mathbf{x}(A\mathbf{xc} \to B\mathbf{xc})$. We have $\alpha \Vdash \forall\mathbf{x}(A\mathbf{xc} \to B\mathbf{xc})$ exactly when for all $\beta \sqsupset \alpha$ and $\mathbf{d} \in M_\beta$ we have $\beta \Vdash (A\mathbf{dc} \wedge \mathrm{E}\,\mathbf{d} \vdash B\mathbf{dc})$. In terms of the proof interpretation, at node $\beta \sqsupset \alpha$ construction σ at stage α for universal implication sentence $\forall\mathbf{x}(A\mathbf{x} \to B\mathbf{x})$ has been sufficiently completed to obtain a proof $(A\mathbf{x}, \pi\mathbf{x}, B\mathbf{x})$. In terms of the proof interpretation, at node $\beta \sqsupseteq \alpha$ with $\mathbf{c} \in M_\beta$, construction $\sigma\mathbf{y}$ at stage α for universal implication formula $\forall\mathbf{x}(A\mathbf{xy} \to B\mathbf{xy})$ implies a construction $\sigma\mathbf{c}$ for universal implication sentence $\forall\mathbf{x}(A\mathbf{xc} \to B\mathbf{xc})$. Are transitive Kripke models insufficient? Yes. Are they evidence of non-provability? Also yes. If one is willing to accept transitive Kripke models and the completeness theorem as evidence, then BQC is complete. We give a non-trivial illustration. The following is the case:

Not a general rule: $$\frac{C \vdash_{y\mathbf{z},\vec{D}} \forall\mathbf{x}((A \wedge \mathrm{E}\,y) \to B)}{C \vdash_{\mathbf{z},\vec{D}} \forall\mathbf{x}y(A \to B)}$$ y not free in $C, \vec{D}$

An explanation of why this is not a general rule may benefit from the following observation over Basic Propositional Logic:

Not a general rule: $$\frac{A \vdash \top \to B}{\top \vdash A \to B}$$

For consider a transitive Kripke model with nodes $\alpha \sqsubset \beta$ and β irreflexive. Let all nodes $\delta \sqsupseteq \alpha$ be such that $\delta = \alpha$ or $\delta \sqsupseteq \beta$. Set $\delta \Vdash A$ exactly when $\delta \sqsupseteq \beta$ and $\delta \Vdash B$ exactly when $\delta \sqsupset \beta$. So $\alpha \Vdash (A \vdash (\top \to B))$ and $\alpha \nVdash (\top \vdash (A \to B))$. A Kripke model sketches a possible constructive situation. We understand this model as an illustration of the significance of the order by which mathematics advances. In the given Kripke model mathematics has to advance twice. First we may advance and get a proof of A from which a proof of $\top \to B$ follows, say construction σ. Second we advance by sufficiently completing σ and get a proof of B. The bottom sequent of the rule doesn't hold. In the Kripke model it implies that if mathematics advances we would get a proof (A, π, B) besides a proof of A, hence by composition (not modus ponens) a premature claim of a proof of B. So the order of advancement matters. As to the earlier rule: On one hand the rule holds in the special case when C equals $\top$. On the other hand we even have the special case:

$$\text{Not a general rule:} \quad \frac{C \ \vdash_y \ (\mathrm{E}\, y \to By)}{C \ \vdash \ \forall y(\top \to By)} \quad y \text{ not free in } C$$

For consider a transitive Kripke model with nodes $\alpha \sqsubset \beta \sqsubset \gamma$, and β and γ irreflexive. All other nodes $\delta \sqsupseteq \alpha$ satisfy $\delta \sqsupset \gamma$. Set $\delta \Vdash C$ exactly when $\delta \sqsupseteq \beta$. Let c be a new description of an element above γ satisfying $\gamma \Vdash \mathrm{E}\, c$. Set $\delta \Vdash Bc$ exactly when $\delta \sqsupset \gamma$. There are no other domain elements d above any node for which $\mathrm{E}\, d$ holds. Now $\alpha \Vdash_y \ (C \vdash_y (\mathrm{E}\, y \to By))$ and $\alpha \nVdash \ (C \vdash \forall y(\top \to By))$. A Kripke model sketches a possible constructive situation. In the given Kripke model mathematics has to advance three times. First we may advance and get a proof of C from which a proof πy of $\mathrm{E}\, y \to By$ follows. In the Kripke model, construction πy need not reveal new methods of reasoning on its own even after completion. Construction πy only implies a further construction of a proof $(\mathrm{E}\, d, \xi d, Bd)$ after a description d is provided. Composition πd is a construction for a proof $(\mathrm{E}\, d, \xi d, Bd)$ rather than a composition of proofs, which would have been a proof. In the model a proof $(\mathrm{E}\, d, \rho d, Bd)$ can only be obtained by sufficiently completing construction πd. Second we will advance and get a description (in essence a construction) c and a proof of $\mathrm{E}\, c$. This gives us proof construction πc of $\mathrm{E}\, c \to Bc$, so also a proof of $\top \to Bc$. Finally mathematics has to advance a third time by sufficiently completing πc to obtain a proof of Bc. The bottom sequent doesn't hold. It implies that if mathematics advances we get a proof of C from which a proof σ of $\forall y(\top \to By)$ follows. Construction σ is independent of y, so by sufficient completion of σ mathematics advances with a proof $(\mathrm{E}\, y, \tau y, By)$. With the next advance with a description of c and proof of $\mathrm{E}\, c$ we compose constructions and get $(\mathrm{E}\, c, \tau c, Bc)$, which with another proof composition allows us to claim Bc prematurely.

6 Conclusion and Remarks

A modest grasp of what is constructive mathematics suffices to uniquely identify constructive predicate logic as BQC of [Ru98].

Although a comprehensive understanding of the meaning of 'constructive' may not exist, there is the expectation of computability in some strong form. Kleene's realizability and Martin Hyland's effective topos are examples that add numerical meaning to intuitionistic logic and arithmetic. Addition of computable meaning is evidence for their constructive nature, but is not sufficient to make them constructive.

Gödel argued in [Go33] that Heyting Arithmetic HA, the intuitionistic version of Peano Arithmetic PA, includes principles that go beyond computability, in particular in its use of negation as a special case of implication. As an alternative, his Dialectica interpretation of 1941–1972, named after [Go58], builds a hierarchy of types starting from primitive recursive arithmetic PRA in an attempt to avoid HA and give numerical meaning to a significant part of constructive mathematics. There are different views about the level of success of this attempt. A generalization of Markov's Principle holds under the Dialectica interpretation. On one hand this may be understood as illustration of a "higher degree of constructivity", as Gödel wrote on [Go72, Go90, page 276]. On the other hand fewer objects satisfy more theorems, so this may be understood as too strong a restriction on which objects are permitted when we look for a constructive logic which expresses very general mathematical theorems about sets and their subsets. William Tait writes on [Ta06, page 217]:

> The proper locus of constructivity is in our *reasoning* and, in particular, in our reasoning about numbers and functions, not in the concepts of number and function.

We may extend the system to Martin-Löf's intuitionistic theory of types of [ML72, ML80], where propositions are considered as types as in the Curry-Howard propositions-as-types interpretation of [Ho69]. The canonical proofs are singled out as the defining notion. This improperly limits our reasoning so much that modus ponens holds. Even in an appropriate theory of constructions and proofs for Basic logic we can show that a proof of $\vdash \ A \to B$ implies that there is a canonical proof, which in turn implies that there is a proof of $A \vdash B$. Nevertheless we cannot in general conclude $C \wedge A \ \vdash \ B$ from $C \ \vdash \ A \to B$. With a growing accumulation of proofs we cannot generalize weak modus ponens to full modus ponens.

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