THE UNINTENDED INTERPRETATIONS OF INTUITIONISTIC LOGIC

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Abstract. We present an overview of the unintended interpretations of intuitionistic logic that arose after Heyting formalized the "observed regularities" in the use of formal parts of language, in particular, first-order logic and Heyting Arithmetic. We include unintended interpretations of some mild variations on "official" intuitionism, such as intuitionistic type theories with full comprehension and higher order logic without choice principles or not satisfying the right choice sequence properties. We conclude with remarks on the quest for a correct interpretation of intuitionistic logic.

§1. The Origins of Intuitionistic Logic

Intuitionism was more than twenty years old before A. Heyting produced the first complete axiomatizations for intuitionistic propositional and predicate logic: according to L. E. J. Brouwer, the founder of intuitionism, logic is secondary to mathematics. Some of Brouwer's papers even suggest that formalization cannot be useful to intuitionism. One may wonder, then, whether intuitionistic logic should itself be regarded as an unintended interpretation of intuitionistic mathematics.

I will not discuss Brouwer's ideas in detail (on this, see [Brouwer 1975], [Heyting 1934, 1956]), but some aspects of his philosophy need to be highlighted here. According to Brouwer mathematics is an activity of the human mind, a product of languageless thought. One cannot be certain that language is a perfect reflection of this mental activity. This makes language an uncertain medium (see [van Stigt 1982] for more details on Brouwer's ideas about language).

In "De onbetrouwbaarheid der logische principes" ([Brouwer 1981], pp. 253–259; for English translations of Brouwer's work on intuitionism, see [Brouwer 1975]) Brouwer argues that logical principles should not guide but describe regularities that are observed in mathematical practice. The Principle of Excluded Third, $A \vee \neg A$, is an example of a logical principle that has become a guide for mathematical practice instead of simply describing it: the Principle of Excluded Third is observed in verifiable "finite" situations and generalized to a rule of mathematics. But according to Brouwer mathematics is not an experimental science, in which one only has to repeat an experiment sufficiently often to establish a law, so the Principle of Excluded Third should be discarded.

All his life Brouwer avoided the use of a formal language or logic, perhaps because of its unreliability, perhaps because of his personal style (see [Brouwer 1981a], p. xi). This does not imply that he did not believe in the possibility of a useful place for logic in intuitionistic mathematics, but rather that Brouwer would not himself resort to a formal language. This attitude was detrimental to the development of intuitionistic logic: it was not until 1923 that Brouwer discovered the equivalence in intuitionistic mathematics of triple negation and single negation [Brouwer 1925].

While Brouwer may have been uncompromising with respect to his philosophy, his mathematical and philosophical talent was understood and appreciated by his thesis adviser D. J. Korteweg. In 1908 Korteweg advised Brouwer, after Brouwer completed his thesis, to devote some time to "proper" mathematics, as opposed to foundations, so as to earn recognition and become eligible for a university position. This initiated Brouwer's brief career as a topologist. Between 1909 and his appointment as Professor at the University of Amsterdam in 1912 Brouwer developed the technique of triangulation, showed the invariance of dimension, and proved his fixed-point theorem ([Brouwer 1975, 1975a]).

Brouwer's ideas about language did not prevent others from considering formalizations of parts of intuitionism. A. N. Kolmogorov [Kolmogorov 1925] gave an incomplete description of first-order predicate logic. Of particular interest is his description of the double negation translation. Although this is sketchy in some parts, it is fair to say that Kolmogorov anticipated Gödel's translation from classical to intuitionistic formal systems (§4, [Gödel 1933]). V. Glivenko [Glivenko 1928] described a fragment of intuitionistic propositional logic in order to establish the double negation of the Principle of Excluded Third. This was in reply to M. Barzin and A. Errera's 1927 paper, in which they attempted to prove Brouwer's mathematics inconsistent. Glivenko's 1929 paper appeared after he had seen Heyting's formalization of intuitionistic logic. In this paper Glivenko showed that the double negation of each classically derivable propositional statement is intuitionistically derivable. This result can not be extended to first-order logic, as $\neg\forall x(P(x)\vee\neg P(x))$ cannot be contradicted in intuitionistic logic for unary atomic predicates P.

It took someone other than Brouwer to provide the first complete axiomatization of first-order intuitionistic logic. Heyting, a former student of Brouwer, wrote his Ph.D. thesis in 1925 on an intuitionistic axiomatization of projective geometry, the first substantial contribution to the intuitionistic program not from Brouwer himself [Troelstra 1981]. In 1927 the Dutch Mathematical Society published a prize question that included the quest for a formalization of intuitionistic mathematics. Heyting wrote an essay on the topic, for which he was awarded the prize in 1928. An expanded version appeared in [Heyting 1930, 1930a, 1930b]. It included an axiomatization of intuitionistic first-order logic (using a version of equality between partial terms, thereby foreshadowing the existence predicate of D. S. Scott [Scott 1979]); Heyting Arithmetic, HA (the intuitionistic equivalent of Peano Arithmetic, PA); and analysis (the theory of choice sequences), though this last axiomatization was not complete.

In modern notation, using sequents, Heyting's axiomatization of intuitionistic predicate logic can be stated as follows. There are three logical axiom schemas and two axiom schemas for equality. The closure rules are: a thin horizontal line means that if the sequents above the line hold, then so do the ones below the line; a fat line means the same as a thin line, but in either direction.

$$A \vdash A$$

$$\frac{A \vdash B \quad B \vdash C}{A \vdash C}$$

$$A \vdash \top \qquad\qquad \bot \vdash A$$

$$\frac{A \vdash B \quad A \vdash C}{A \vdash B \wedge C} \qquad \frac{B \vdash A \quad C \vdash A}{B \vee C \vdash A}$$

$$\frac{A \wedge B \vdash C}{A \vdash B \to C}$$

$$\frac{A \vdash Bx}{A \vdash Bt}\ \dagger$$

$$\frac{A \vdash Bx}{A \vdash \forall x Bx}\ \ddagger \qquad \frac{Bx \vdash A}{\exists x Bx \vdash A}\ \ddagger$$

$$\top \vdash x = x$$

$$x = y \vdash Ax \to Ay \ *$$

In case $\dagger$ the variable x is not free in A and the term t does not contain a variable bound by a quantifier of B; in cases $\ddagger$ the variable x is not free in A; and in case $*$ the variable y is not bound by a quantifier of A.

Let Γ be a set of formulas, sequents, and rules and let σ be a formula. We write $\Gamma \vdash \sigma$, Γ *proves* σ, if there exists a finite subset $\Delta \subseteq \Gamma \cup \{\top\}$ such that $\bigwedge_{\delta \in \Delta} \delta \vdash \sigma$ is a sequent derivable from the system above plus the sequents and rules of Γ. We use $\vdash \sigma$ as an abbreviation for $\top \vdash \sigma$, $\neg\sigma$ as an abbreviation for $\sigma \to \bot$, and $\sigma \leftrightarrow \tau$ as an abbreviation for $(\sigma \to \tau) \wedge (\tau \to \sigma)$. If we add the axiom schema $\vdash A \vee \neg A$ to the system above of intuitionistic logic, we obtain an axiomatization for classical first-order logic.

In the language of HA we have the usual binary function symbols $+$ and $\cdot$, the unary function symbol S, and the constant symbol 0. The axiom system of HA has the following non-logical axiom schemas and rule:

$$Sx = 0 \vdash \bot$$

$$Sx = Sy \vdash x = y$$

$$\vdash x + 0 = x$$

$$\vdash x + Sy = S(x + y)$$

$$\vdash x \cdot 0 = 0$$

$$\vdash x \cdot Sy = (x \cdot y) + x$$

$$\frac{A(x) \vdash A(Sx)}{A(0) \vdash A(x)}$$

where x is not free in $A(0)$ in the rule above. The system HA is strong enough to prove essentially all number theoretic results we find in any text on number theory. The main exceptions involve theorems related to incompleteness proofs of PA and statements that are true in PA only because of their logical form, like $\vdash \omega \vee \neg\omega$ for some undecidable statement ω. In fact, if $HA \vdash \sigma \vee \tau$, then $HA \vdash \sigma$ or $HA \vdash \tau$. Similarly, if $HA \vdash \exists x \sigma x$, then $HA \vdash \sigma m$ for some natural number m.

The systems of classical mathematics and intuitionistic mathematics diverge more noticeably when we consider the theory of real numbers or abstract mathematics.

In a letter to O. Becker in 1933 Heyting described how he came to his axiomatization essentially by going through the axioms and rules of the Principia Mathematica [Whitehead, Russell 1925] and making a new system of axioms and rules out of the acceptable axioms. (The papers [Troelstra 1978] and [Troelstra 1981] discuss in more detail the period around the formalization of intuitionistic logic.)

Like Brouwer, Heyting considered logic with mixed feelings. Heyting feared that his formalization would divert attention from the underlying issues (see [Heyting 1930, 1930a, 1930b], or the discussion of these papers by A. S. Troelstra in [Troelstra 1978] and [Troelstra 1981]). Later he even expressed disappointment in that his papers [Heyting 1930, 1930a, 1930b] had distracted the attention from the underlying ideas to the formal systems themselves. Those papers, then, do not represent a divergence between Heyting's ideas and Brouwer's. On the contrary, Brouwer clearly appreciated Heyting's contributions to clarify intuitionism [Troelstra 1978].

Alongside intuitionism other schools of constructive mathematics developed. Most well–known among these are finitism [Hilbert, Bernays 1934], Markov (recursive) constructivism ([Šanin 1958], [Markov 1962]), and Bishop's constructivism [Bishop 1967]. A discussion of these can be found in [Troelstra 1977]. Here it suffices to note that all these constructive schools settled on the same first-order logic, by means of different philosophies. Most of the essential differences involve second- or higher-order mathematical questions. Differences also occur at the level of arithmetic; in particular, the Markov school allows the so-called Markov Principle to extend HA:

$$\forall n(An \vee \neg An) \wedge \neg\neg\exists n An \vdash \exists n An,$$

where A is a formula of arithmetic. The corresponding *rule* is weaker and already holds in HA:

$$\frac{HA \vdash \forall n(An \vee \neg An) \wedge \neg\neg\exists n An}{HA \vdash \exists n An}.$$

§2. Interpretations for Propositional Logic

The main impact of Heyting's formalization of intuitionistic logic was its availability to a much wider audience of mathematicians and logicians. For the first time non-intuitionists could get a hold on intuitionism. This brought some vital intellectual blood into the development of intuitionism; before 1940 there were very few intuitionists, even in the Netherlands [Troelstra 1981]. Now non-intuitionists began to take steps to relate intuitionistic logic with other concepts, in particular, the theory of recursive functions and model theory. An early example of this is [Gödel 1932] (see [Gödel 1986] for all early work of Gödel). In this paper Gödel essentially constructed a countable properly descending sequence of logics T_i between intuitionistic propositional calculus and the classical propositional calculus such that each T_i is valid on just those linearly ordered Heyting algebras of length at most $i + 1$.

The first completeness theorem for the intuitionistic propositional calculus came from M. H. Stone [Stone 1937] and A. Tarski [Tarski 1938] in 1937–38. They discovered that topological spaces form a complete set of models for the intuitionistic propositional calculus in the same way that Boolean spaces do for the classical propositional calculus: the completeness theorem for the classical propositional calculus based on Boolean spaces X works by assigning to each atom a clopen (closed and open) subset of X and by interpreting disjunction, conjunction, and negation as set theoretic union, intersection, and relative complement. For intuitionistic propositional logic Stone and Tarski obtained a completeness theorem for topological spaces by assigning open sets to atoms and by interpreting disjunction, conjunction, and implication as in the classical case, except that when a resulting set was not open, it had to be replaced by its interior. See the remarks below on the relationship of this interpretation to an interpretation for $S4$.

The significance of Tarski and Stone's interpretation (geometrization of logic) is clear from our current perspective. It was too early for further development, however, as category theory did not appear until the 1940's, and sheaf theory had to wait nearly until the 1950's.

In [Gödel 1933a] Gödel embedded intuitionistic propositional logic into Lewis's modal propositional logic $S4$. The system $S4$ is a propositional logic with a modal operator, $\Box$. For the logical constants $\top$ and $\bot$ and the logical operators $\wedge$, $\vee$, $\rightarrow$, and $\neg$ we take the rules and axioms of classical propositional logic. The modal operator $\Box$, "provable", satisfies all substitution instances of the axioms

$$\begin{aligned} \Box A &\vdash A, \\ \Box(A \rightarrow B) &\vdash \Box A \rightarrow \Box B, \\ \Box A &\vdash \Box\Box A, \end{aligned}$$

and the rule

$$\frac{\vdash A}{\vdash \Box A}.$$

In [Gödel 1933a] we actually find two closely related translations. One of Gödel's embeddings $A \mapsto A'$ is equivalent to the following inductively defined translation:

$$\begin{aligned} \top' &= \Box\top; \\ \bot' &= \Box\bot; \\ p' &= \Box p, \qquad p \text{ an atom}; \\ (A \wedge B)' &= \Box(A' \wedge B'); \\ (A \vee B)' &= \Box(A' \vee B'); \quad \text{and} \\ (A \rightarrow B)' &= \Box(A' \rightarrow B'). \end{aligned}$$

We can easily show that some of the $\Box$'s in the translation above are redundant; $S4 \vdash A' \leftrightarrow \Box(A')$. For this interpretation

$$\vdash \sigma \quad \text{if and only if} \quad S4 \vdash \sigma'.$$

Gödel proved only one direction: if σ holds intuitionistically, then its translation σ' holds in $S4$. The reverse implication was established by J. C. C. McKinsey and A. Tarski [McKinsey, Tarski 1948].

A topological model for $S4$ consists of a set X provided with a topology. We assign a subset $[\![P]\!] \subseteq X$ to each atom P and extend the interpretation inductively to all propositions of the language by defining

$$\begin{aligned} [\![\top]\!] &= X; \\ [\![\bot]\!] &= \emptyset; \\ [\![A \wedge B]\!] &= [\![A]\!] \cap [\![B]\!]; \end{aligned}$$

$$
\begin{aligned}
[\![A \vee B]\!] &= [\![A]\!] \cup [\![B]\!]; \\
[\![\neg A]\!] &= [\![A]\!]^c, \qquad \text{where } ^c \text{ stands for complement}; \\
[\![A \to B]\!] &= [\![A]\!]^c \cup [\![B]\!]; \quad \text{and} \\
[\![\Box A]\!] &= \operatorname{Int}[\![A]\!], \qquad \text{where Int stands for interior.}
\end{aligned}
$$

Using Gödel's interpretation of intuitionistic logic into $S4$, we obtain the Stone-Tarski interpretation for intuitionistic logic: a topological model for intuitionistic propositional logic consists of a set X provided with a topology. We assign an open subset $[\![P]\!] \subseteq X$ to each atom P. The interpretation is extended inductively to all propositions of the language by defining

$$
\begin{aligned}
[\![\top]\!] &= X; \\
[\![\bot]\!] &= \emptyset; \\
[\![A \wedge B]\!] &= [\![A]\!] \cap [\![B]\!]; \\
[\![A \vee B]\!] &= [\![A]\!] \cup [\![B]\!]; \\
[\![\neg A]\!] &= \operatorname{Int}([\![A]\!]^c); \quad \text{and} \\
[\![A \to B]\!] &= \operatorname{Int}([\![A]\!]^c \cup [\![B]\!]).
\end{aligned}
$$

We write $(X, [\![\cdot]\!]) \models \varphi$ if $[\![\varphi]\!] = X$. The completeness theorem is: for all sets of propositions $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash \varphi \iff \text{for all } (X, [\![\cdot]\!]), \text{ if } (X, [\![\cdot]\!]) \models \gamma \text{ for all } \gamma \in \Gamma, \text{ then } (X, [\![\cdot]\!]) \models \varphi.$$

Several mild variations on the equivalence between $S4$ and Gödel's translation have been discovered since 1933. There is, however, another modal logic of note, which is based on the provability operator in PA. This system, called G, was discovered by R. M. Solovay in 1976 to be sufficient to characterize a certain form of provability in PA. The system G distinguishes itself from $S4$ by having all substitution instances of the axioms

$$
\begin{aligned}
&\Box(A \to B) \vdash \Box A \to \Box B, \\
&\Box(\Box A \to A) \vdash \Box A \qquad \text{(Löb's rule)},
\end{aligned}
$$

and the rule

$$\frac{\vdash A}{\vdash \Box A}$$

for the modal operator $\Box$. The axiom schema $\Box A \vdash \Box\Box A$ is derivable in G. Note that $S4$ and G are relatively inconsistent.

The formulas of the modal language of G can be embedded into the language of number theory as follows. Let $Prov(\ulcorner \sigma \urcorner)$ be the proof predicate of PA. Each map Φ that maps the atoms p of the modal language to sentences Φp of the language of PA can be extended to a map of all formulas by induction: $\Phi\top = \top$, $\Phi\bot = \bot$, $\Phi(A \circ B) = \Phi A \circ \Phi B$ for $\circ \in \{\wedge, \vee, \to\}$, $\Phi\neg A = \neg\Phi A$, and $\Phi(\Box A) = Prov(\ulcorner \Phi A \urcorner)$. Then Solovay's result says

$$G \vdash \sigma \quad \text{if and only if} \quad PA \vdash \Phi\sigma \text{ for all } \Phi.$$

In 1978 R. Goldblatt showed how to interpret intuitionistic propositional logic in G. Combine the translation $A \mapsto A'$ with the translation $A \mapsto A^\diamond$ from the modal language to itself, defined inductively by $p^\diamond = p$, $\top^\diamond = \top$, $\bot^\diamond = \bot$, $(A \circ B)^\diamond = A^\diamond \circ B^\diamond$ for $\circ \in \{\wedge, \vee, \rightarrow\}$, $(\neg A)^\diamond = \neg A^\diamond$, and $(\Box A)^\diamond = A^\diamond \wedge \Box(A^\diamond)$. Goldblatt's result can then be stated as

$$\vdash \sigma \quad \text{if and only if} \quad G \vdash (\sigma')^\diamond.$$

In his paper Goldblatt mentioned an earlier proof by A. Kuznetsov and A. Muzavitski.

Combining the results of Solovay and Goldblatt, we obtain

$$\vdash \sigma \quad \text{if and only if} \quad PA \vdash \Phi((\sigma')^\diamond) \text{ for all } \Phi.$$

Most interpretations for propositional calculus were also extended to first-order logic interpretations (see §5).

§3. Realizability

By the early 1930's there were two theories that dealt with the notion of constructive process: intuitionism and recursion theory.

In 1931 Heyting developed the proof-interpretation that accompanies his formalization of intuitionism. In [Heyting 1934] we find both Heyting's interpretation and Kolmogorov's problem-interpretation of 1932, while in [Hilbert, Bernays 1934] the concept of "incomplete communication" of a constructive statement is discussed. Brouwer himself was vague about the interpretation of the logical operators.

On the other hand, the notion of a recursive function was, by Church's thesis (1936), considered to be equivalent to that of an effectively computable function.

S. C. Kleene was the first to seriously consider the possibility of a more precise connection between the two, in particular, between HA and recursive function theory [Kleene 1973]. When an intuitionist claims $\exists n \sigma n$ there should be an effective process p to find n. When an intuitionist claims $\forall m \exists n \sigma(m, n)$ there must be a construction p such that for all numbers m, pm is an effective process for finding an n such that $\sigma(m, n)$. Kleene conjectured in 1940 that this p should embrace the existence of a recursive function f such that $\sigma(m, fm)$ holds for all m. This was not at all obvious at the time [Kleene 1973].

In [Hilbert, Bernays 1934] it was expounded that an intuitionistic statement like $\exists m \sigma m$ is an incomplete communication of a more involved statement giving m such that σm. In the same way a statement like $\sigma \vee \tau$ is an incomplete communication of a more involved statement either establishing σ or establishing τ. Kleene's idea was to somehow add this missing information to statements so that when he arrived at expressions like $\forall m \exists n \sigma(m, n)$ the added information would include the requested recursive function. Since formulas in HA are defined inductively by complexity, adding the missing information would have to be done by induction on the complexity of formulas. In early 1941 Kleene finally succeeded in bringing this idea to fruition, finding a version of it which encoded the missing information in natural numbers: number realizability [Kleene 1945].

In number realizability missing information, encoded in numbers, is added in such a way that a formula σ is realizable by a number e, written $e\mathbf{r}\sigma$, if it is derivable in HA. The intended meaning of $e\mathbf{r}\sigma$ is that the number e encodes information "missing" from the communication that the formula σ is true intuitionistically. It is defined by induction on the complexity of formulas of the language of HA. This notion of realizability did not follow Heyting's proof-interpretation ([Heyting

1934], §7), which indicated a certain liberty in defining variations and extensions of realizability, a fact already employed in [Kleene 1945].

According to the proof-interpretation an implication $\sigma \to \tau$ is established only if we provide a construction C that converts a proof of σ into a proof of τ. The Hilbert–Bernays version distinguishes itself by requiring that the construction C convert information establishing σ into information establishing τ. A key idea of number realizability was to use partial recursive functions instead of total recursive functions, which helped in extending the definition of realizability through implication. A total construction replies to all input. If the input is not information establishing σ, the reply may be meaningless. A partial construction need only provide information (establishing τ) from information establishing σ; on other data it may be undefined. It is not clear whether before 1941 anyone else was aware of the importance of this distinction.

Let $\langle x, y\rangle$ represent a recursive pairing function over HA, with recursive projection functions $p_1 x$ and $p_2 x$. Thus, $x = \langle p_1 x, p_2 x\rangle$, $p_1\langle x, y\rangle = x$, and $p_2\langle x, y\rangle = y$. Let $\{\cdot\}\cdot$ represent the Kleene bracket expression, where $\{x\}y$ is the result of applying the xth partial recursive function to the number y; write $!\{x\}y$ to indicate that $\{x\}y$ is defined. Since $\{x\}y$ is a partial function, it is more natural to describe it by a recursive relation. This is essentially Kleene's T-predicate. For our purposes we use the existence of a ternary relation $T(x, y, z)$ such that $T(x, y, z) \leftrightarrow \{x\}y = z$. All expressions mentioned above are expressible in the language of HA.

Number realizability is a translation $A \mapsto x\mathbf{r}A$ for formulas not containing the variable x, which is defined inductively by:

$$\begin{aligned}
x\mathbf{r}\top &= \top;\\
x\mathbf{r}\bot &= \bot;\\
x\mathbf{r}(t = u) &= t = u, \qquad t,u \text{ terms};\\
x\mathbf{r}(A \wedge B) &= p_1 x\mathbf{r}A \wedge p_2 x\mathbf{r}B;\\
x\mathbf{r}(A \vee B) &= (p_1 x = 0 \to p_2 x\mathbf{r}A) \wedge (p_1 x \neq 0 \to p_2 x\mathbf{r}B);\\
x\mathbf{r}(A \to B) &= \forall y(y\mathbf{r}A \to !\{x\}y \wedge \{x\}y\mathbf{r}B);\\
x\mathbf{r}(\exists y Ay) &= p_2 x\mathbf{r}A(p_1 x); \quad \text{and}\\
x\mathbf{r}(\forall y Ay) &= \forall y(!\{x\}y \wedge \{x\}y\mathbf{r}Ay).
\end{aligned}$$

A formula is called *almost negative* if it is built up from formulas of the form $\exists x(t = u)$ using $\wedge$, $\to$, and $\forall$. The added information in the construction of the realizability translation embodies the assumption that all functions are recursive. This idea can be formalized by the axiom schema ECT_0, the *extended Church's thesis*:

$$\forall x(Ax \to \exists y B(x, y)) \vdash \exists z \forall x(Ax \to !\{z\}x \wedge B(x, \{z\}x)),$$

where A is an almost negative formula. The study of Kleene's number realizability culminated in the following results of Troelstra ([Troelstra 1973, p. 196]:

$$HA + ECT_0 \vdash A \leftrightarrow \exists x(x\mathbf{r}A)$$

and

$$HA + ECT_0 \vdash A \text{ if and only if } HA \vdash \exists x(x\mathbf{r}A).$$

So if $HA \vdash \forall x \exists y \sigma(x,y)$, then there is a number e such that $HA \vdash e\mathbf{r}\forall x \exists y \sigma(x,y)$. Using the translation above, this implies

$$HA \vdash \forall x (!\{e\}x \wedge p_2(\{e\}x)\mathbf{r}\sigma(x, p_1(\{e\}x))).$$

So $f(x) = p_1(\{e\}x)$ is the recursive function sought after for HA extended with ECT_0:

$$HA + ECT_0 \vdash \forall x \sigma(x, f(x)).$$

Number realizability did explicate the Hilbert-Bernays interpretation of incomplete information, but it did not give a recursive witness for $\forall x \exists y \sigma(x,y)$–sentences over HA simpliciter. To make the method above work for HA alone, Kleene introduced a variation of number realizability consisting of a translation $A \mapsto x\mathbf{q}A$ for formulas not containing the variable x, defined inductively by:

$$\begin{aligned}
x\mathbf{q}\top &= \top; \\
x\mathbf{q}\bot &= \bot; \\
x\mathbf{q}(t = u) &= t = u, \qquad t{,}u \text{ terms}; \\
x\mathbf{q}(A \wedge B) &= p_1 x\mathbf{q}A \wedge p_2 x\mathbf{q}B; \\
x\mathbf{q}(A \vee B) &= (p_1 x = 0 \to A \wedge p_2 x\mathbf{q}A) \wedge (p_1 x \neq 0 \to B \wedge p_2 x\mathbf{q}B); \\
x\mathbf{q}(A \to B) &= \forall y((A \wedge y\mathbf{q}A) \to !\{x\}y \wedge \{x\}y\mathbf{q}B); \\
x\mathbf{q}(\exists y A y) &= A(p_1 x) \wedge p_2 x\mathbf{q}A(p_1 x); \quad \text{and} \\
x\mathbf{q}(\forall y A y) &= \forall y(!\{x\}y \wedge \{x\}y\mathbf{q}Ay).
\end{aligned}$$

Using $\mathbf{q}$–realizability, we can show

$$HA \vdash \forall x (Ax \to \exists y B(x,y))$$

implies

$$HA \vdash \forall x (Ax \to !\{e\}x \wedge B(x, \{e\}x)) \text{ for some } e,$$

where A is almost negative. Then $f(x) = \{e\}x$ is the function that Kleene conjectured in 1940.

Kleene's original conjecture went beyond HA to claim that if in any intuitionistic system that includes HA we prove $\vdash \forall x \exists y \sigma(x,y)$, then there should be a recursive function f such that $\vdash \forall x \sigma(x, fx)$. For that reason Kleene developed a formal system for Heyting's Analysis, the theory of choice sequences, and in 1957 extended realizability to it (see [Kleene 1973] for further references).

For later work on and applications of realizability, see [Troelstra 1973], [Beeson 1985].

The connection above between intuitionistic arithmetic and recursion theory suggests that there may be connections between computer science and intuitionism as well.

The computer science aspect of recursion theory focuses on complexity theory rather than on the class of all general recursive functions. Following a conjecture of S. Cook, S. Buss found subsystems of HA that allowed for an analog of Kleene's realizability, except that instead of representing all general recursive functions, the representable functions are the ones restricted to subclasses of the polynomial-complexity hierarchy.

Let $\Sigma_0^p = \Pi_0^p$, Σ_1^p, Π_1^p, Σ_2^p, Π_2^p, $\cdots$ be the Meyer–Stockmeyer polynomial hierarchy of predicates [Stockmeyer 1976], where $NP = \Sigma_1^p$, and co-$NP = \Pi_1^p$. Define $\Box_i^p = PTC(\Sigma_{i-1}^p)$, the functions computed by a polynomial time Turing machine with an oracle for a Σ_{i-1}^p-predicate. The $0, 1$-valued functions of $\Box_i^p$ form the class Δ_i^p. Note that $P = \Delta_1^p$ is the class of polynomial complexity predicates, and $\Box_1^p$ is the class of polynomial complexity functions.

Based on earlier work [Buss 1985], Buss considered subsystems $IS_2^1 \subseteq IS_2^2 \subseteq IS_2^3 \subseteq \cdots$ of HA, defined as follows: let $|x|$ be the length of the number x written in binary form, that is, $|x| = \lceil log_2(x+1) \rceil$, and let $\lfloor \frac{x}{2} \rfloor$ be division of x by 2, rounded down. A quantifier is *bounded* if it occurs in the form $\forall x \leq t$ or $\exists x \leq t$ for some term t. A quantifier is *sharply bounded* if it occurs in the form $\forall x \leq |t|$ or $\exists x \leq |t|$. A formula is of bounded complexity Σ_i^b if it is equivalent to a formula in prenex form with i bounded-quantifier alternations with Σ on the outside, ignoring sharply bounded quantifiers. A formula is of hereditarily bounded complexity $H\Sigma_i^b$ if all its subformulas are also of complexity Σ_i^b. The systems IS_2^i are axiomatized by basic sets of axioms (depending on i) and the following induction schema for $H\Sigma_i^b$ formulas A:

$$A\lfloor \frac{x}{2} \rfloor \to Ax \vdash A(0) \to \forall x A x.$$

Then the systems IS_2^i satisfy ([Buss 1985a]):

$$IS_2^i \vdash \forall x_1 ... x_n \exists y A(x_1, ..., x_n, y)$$

implies

there exists a $\Box_i^p$-function $f : \mathbf{N}^n \to \mathbf{N}$ such that

$A(m_1, ..., m_n, f(m_1, ..., m_n))$ is valid in $\mathbf{N}$, for all $m_1, ..., m_n$.

Conversely, for each $\Box_i^p$-function f there is a formula $A(\mathbf{x}, y)$ such that for all $\mathbf{m} \in \mathbf{N}^n$, $A(\mathbf{m}, f\mathbf{m})$ is valid in $\mathbf{N}$, and $IS_2^i \vdash \forall \mathbf{x} \exists y A(\mathbf{x}, y)$.

§4. The Double Negation Translation, and the Dialectica Interpretation

Gödel initiated two more unintended interpretations, which for purposes of exposition I will treat in this section. In 1925 Kolmogorov gave an incomplete sketch of the so-called double-negation translation, which essentially consisted of doubly negating each subformula of a formula. This embedded the classical predicate calculus into the intuitionistic calculus. Gödel's translation of [Gödel 1933] in 1933 is an example of a fully developed double negation translation for the case of arithmetic, including a precise translation for the quantifiers.

In modern notation the double negation translation is as follows:

$$\begin{aligned}
\top^\bullet &= \top; \\
\bot^\bullet &= \bot; \\
P^\bullet &= \neg\neg P, \qquad \text{P an atom;} \\
(A \wedge B)^\bullet &= A^\bullet \wedge B^\bullet; \\
(A \vee B)^\bullet &= \neg(\neg A^\bullet \wedge \neg B^\bullet); \\
(A \to B)^\bullet &= A^\bullet \to B^\bullet; \\
(\forall x A)^\bullet &= \forall x A^\bullet; \quad \text{and}
\end{aligned}$$

$$(\exists x A)^\bullet = \neg\forall x \neg A^\bullet.$$

Note that $\vdash \sigma^\bullet \leftrightarrow \neg\neg\sigma^\bullet$. Let C be the axiom schema of Excluded Third, i.e., $\vdash A \vee \neg A$. Then

$$C \vdash \sigma \text{ if and only if } \vdash \sigma^\bullet.$$

Since in HA an atomic formula is equivalent to its double negation, this implies that each formula in HA, built up from the atoms using only negation, conjunction, and universal quantification, is equivalent to its double negation. So from Gödel's translation we obtain the result that for formulas σ of arithmetic, built up from the atoms using only negation, conjunction, and universal quantification,

$$PA \vdash \sigma \text{ if and only if } HA \vdash \sigma.$$

Since in classical logic each first-order formula is equivalent to a formula using only negation, conjunction, and universal quantification, this implies that PA and HA are equiconsistent and that PA is embedded in HA.

The result above was discovered independently by Gentzen and Bernays in 1933.

In 1958 Gödel gave the Dialectica interpretation, in which logical complexity was replaced by the use of higher types. The language $L(\mathbf{N}-\mathbf{HA}^\omega)$ that it was originally designed for is a typed language with equality. The types are defined inductively: there is a bottom type N and for each pair of types r, s an operator type $(r)s$ which is a set of operators from r to s. The axiom of extensionality $\forall fg \in (r)s((\forall x \in r fx = gx) \rightarrow f = g)$ is not assumed from the outset: In the model HRO mentioned below the elements of $(r)s$ are essentially algorithms describing functions from r to s rather than functions themselves. For each type s there are variables x^s, y^s, $\cdots$. New terms can be constructed from old ones by composition, provided that the types match.

The theory $\mathbf{N}-\mathbf{HA}^\omega$ is based on many-sorted intuitionistic predicate logic plus some defining axioms for particular constants: constants $0 \in N$ and $S \in (N)N$, and for all r, s, and t, constants $\Pi_{s,t} \in (s)(t)s$, $\Sigma_{r,s,t} \in ((r)(s)t)((r)s)(r)t$, and $R_s \in (s)((N)(s)s)(N)s$. Subscripts and superscripts will be suppressed when the meaning of the terms is unambiguous. We write xyz as abbreviation for $(xy)z$. For 0 and S we have the usual number theoretic axioms of zero and successor. For Π, Σ, and R, $\Pi xy = x$, $\Sigma xyz = xz(yz)$, $Rxy0 = x$, and $Rxy(Sz) = y(Rxyz)z$. The constants Π and Σ are analogous to the terms $K = \lambda xy.x$ and $S = \lambda xyz.xz(yz)$ of lambda calculus [Barendregt 1984, p. 31], and the elements k and s of a combinatory algebra [Barendregt 1984, p. 90]. The constants R are used to construct the primitive recursive functions ([Troelstra 1973, pp. 51ff], [Barendregt 1984, pp. 568ff]). By using the higher types we actually obtain a larger class of recursive functions.

There are several models for this theory mentioned in [Troelstra 1973, Ch. 2] and an overview in [Troelstra 1977, pp. 1026ff]. One non-trivial example is the model HRO, the *hereditarily recursive operations*. Assign a domain $V_r \subseteq \mathbf{N}$ to each type r such that $V_N = \mathbf{N}$ and $x \in V_{(r)s} \leftrightarrow \forall y \in V_r \exists z \in V_s(\{x\}y = z)$. To make all sets V_r disjoint, replace $n \in V_r$ by $(n, r) \in V_r$. HRO is not extensional, as different elements $(m, r), (n, r) \in V_r$ may define the same partial recursive function $\{m\} = \{n\}$. Application is defined by $(m, (r)s) \cdot (n, r) = (\{m\}n, s)$.

The fundamental idea behind the Dialectica interpretation is the exchange of complexity for higher types. This is illustrated by the replacement of formulas of the form $\forall x \in r \exists y \in s\sigma(x, y)$ by $\exists z \in (r)s\forall x\sigma(x, zx)$. In the presence of sufficiently strong choice principles and Excluded Middle all formulas are equivalent to formulas in prenex form, and repeated application of the replacement above translates a formula A into an equivalent formula of the form $A^D = \exists\mathbf{x}\forall\mathbf{y}A_D(\mathbf{x},\mathbf{y})$ with A_D quantifier-free. The Dialectica interpretation generalizes this translation from classical logic to an intuitionistic theory similar to $\mathbf{N}-\mathbf{HA}^\omega$, called $\mathbf{WE}-\mathbf{HA}^\omega$. The theory $\mathbf{WE}-\mathbf{HA}^\omega$ distinguishes itself from $\mathbf{N}-\mathbf{HA}^\omega$ by the schemas

$$x^{(r)s} = y^{(r)s} \leftrightarrow \forall z^r(x^{(r)s}z^r = y^{(r)s}z^r)$$

and

$$\frac{P \vdash x\mathbf{z} = y\mathbf{z} \quad \vdash Ax}{\vdash Ay},$$

where P is a quantifier-free formula with terms of type N only in which x, y, and $\mathbf{z}$ do not occur. The above schemas make $\mathbf{WE}-\mathbf{HA}^\omega$ *weakly extensional.* For each formula $A\mathbf{z}$ we define two translations, A^D and A_D, where $A^D = \exists\mathbf{x}\forall\mathbf{y}A_D(\mathbf{x},\mathbf{y},\mathbf{z})$ and A_D is quantifier free. We extend the translation inductively as follows (n is a variable over N):

$$A^D = A_D = A, \qquad \text{A an atom.}$$

For the other clauses, let $A^D = \exists\mathbf{x}\forall\mathbf{y}A_D(\mathbf{x},\mathbf{y})$, $B^D = \exists\mathbf{u}\forall\mathbf{v}B_D(\mathbf{u},\mathbf{v})$. Then:

$$\begin{aligned}
(A \wedge B)_D &= A_D \wedge B_D;\\
(A \vee B)_D &= (n = 0 \to A_D) \wedge (n \neq 0 \to B_D);\\
(A \to B)_D &= A_D(\mathbf{x}, \mathbf{Y}\mathbf{x}\mathbf{v}) \to B_D(\mathbf{U}\mathbf{x}, \mathbf{v});\\
(\exists z A(z))_D &= A(z)_D;\\
(\forall z A(z))_D &= A_D(\mathbf{X}z, \mathbf{y}, z);\\
(A \wedge B)^D &= \exists\mathbf{x}\mathbf{u}\forall\mathbf{y}\mathbf{v}(A \wedge B)_D\\
&= \exists\mathbf{x}\mathbf{u}\forall\mathbf{y}\mathbf{v}(A_D \wedge B_D);\\
(A \vee B)^D &= \exists n\mathbf{x}\mathbf{u}\forall\mathbf{y}\mathbf{v}(A \vee B)_D\\
&= \exists n\mathbf{x}\mathbf{u}\forall\mathbf{y}\mathbf{v}(n = 0 \to A_D(\mathbf{x},\mathbf{y})) \wedge (n \neq 0 \to B_D(\mathbf{u},\mathbf{v}));\\
(A \to B)^D &= (\exists\mathbf{x}\forall\mathbf{y}A_D \to \exists\mathbf{u}\forall\mathbf{v}B_D)^D\\
&= (\forall\mathbf{x}\exists\mathbf{u}\forall\mathbf{v}\exists\mathbf{y}(A_D \to B_D))^D\\
&= \exists\mathbf{U}\mathbf{Y}\forall\mathbf{x}\mathbf{v}(A \to B)_D\\
&= \exists\mathbf{U}\mathbf{Y}\forall\mathbf{x}\mathbf{v}(A_D(\mathbf{x}, \mathbf{Y}\mathbf{x}\mathbf{v}) \to B_D(\mathbf{U}\mathbf{x}, \mathbf{v}))\\
&= A_D(\mathbf{x}, \mathbf{Y}\mathbf{x}\mathbf{v}) \to B_D(\mathbf{U}\mathbf{x}, \mathbf{v});\\
(\exists z A(z))^D &= \exists z\mathbf{x}\forall\mathbf{y}(\exists z A(z))_D\\
&= \exists z\mathbf{x}\forall\mathbf{y}A_D(\mathbf{x},\mathbf{y},z); \quad \text{and}\\
(\forall z A(z))^D &= \exists\mathbf{X}\forall z\mathbf{y}(\forall z A(z))_D\\
&= \exists\mathbf{X}\forall z\mathbf{y}A_D(\mathbf{X}z,\mathbf{y},z).
\end{aligned}$$

A^D and A need not be intuitionistically equivalent unless A is of the form $\exists\mathbf{x}\forall\mathbf{y}B$ with B quantifier-free. The Dialectica interpretation has the following properties. If $\mathbf{WE}-\mathbf{HA}^\omega \vdash A$ then $\mathbf{WE}-\mathbf{HA}^\omega \vdash \forall\mathbf{y}A_D(\mathbf{t},\mathbf{y})$ for some sequence of terms $\mathbf{t}$, and

$$\mathbf{WE}-\mathbf{HA}^\omega + S \vdash A \text{ if and only if } \mathbf{WE}-\mathbf{HA}^\omega \vdash A^D$$

for particular extensions S of $\mathbf{WE}-\mathbf{HA}^\omega$ [Troelstra 1977, pp. 1032ff]. See [Troelstra 1973, Ch. 3, § 5] and [Troelstra 1977] for more on the Dialectica interpretation.

§5. Interpretations for Predicate Logic

The extension of the Stone-Tarski models for propositional logic to first-order predicate logic was done by A. Mostowski [Mostowski 1948]. More models for predicate logic soon followed: pseudo–Boolean algebras by J. C. C. McKinsey and Tarski [McKinsey, Tarski 1948] and H. Rasiowa and R. Sikorski [Rasiowa, Sikorski 1953]; Beth models by E. W. Beth [Beth 1956, 1959]; and Kripke models by S. Kripke [Kripke 1965]. Kripke models turned out to be particularly efficient and easy to use (see, for instance, [Smoryński 1973]), because they implicitly use partial elements in their definition. I will give a description of Kripke models below, where I discuss a first-order extension of Lewis's system $S4$, leaving the others as special cases—up to isomorphism—of the sheaf models of §6. There were some techniques to convert a model of one kind to a model of another, but there was no unifying concept of model. In fact, classical set theory seems not to permit the construction of a *natural* unifying concept. The development of a unifying concept had to wait until the further advancement of category theory, particularly topos theory.

Gödel's interpretation of the intuitionistic propositional calculus in Lewis's system $S4$ was extended to an interpretation of the intuitionistic predicate calculus in $QS4$, the first-order generalization of $S4$, independently by Rasiowa and Sikorski [Rasiowa, Sikorski 1953a] and S. Maehara [Maehara 1954]. Kripke models were announced in [Kripke 1959] in 1959 as a specialization of Kripke's models for the system of modal predicate logic $QS4$, in which intuitionistic logic could be embedded.

The system $QS4$ is the first-order extension of the system $S4$, where for terms, equality, and the quantifiers $\exists$ and $\forall$, we have the usual axioms and closure rules:

$$\frac{A \vdash Bx}{A \vdash Bt}\ \dagger$$

$$\frac{A \vdash Bx}{A \vdash \forall x Bx}\ \ddagger \qquad \frac{Bx \vdash A}{\exists x Bx \vdash A}\ \ddagger$$

$$\top \vdash x = x$$

$$x = y \vdash Ax \to Ay\ *$$

In case † the variable x is not free in A and the term t does not contain a variable bound by a quantifier of B; in cases ‡ the variable x is not free in A; and in case $*$ the variable y is not bound by a quantifier of A.

Gödel's embedding $A \mapsto A'$ can be extended to first–order logic by

$$(\forall x Ax)' = \Box(\forall x A'x); \quad \text{and}$$
$$(\exists x Ax)' = \Box(\exists x A'x).$$

Again we have $QS4 \vdash A' \leftrightarrow \Box(A')$, and

$$\vdash \sigma \text{ if and only if } QS4 \vdash \sigma'.$$

Kripke models for $QS4$ are defined by pairs $\mathbf{K} = \langle S, I\rangle$, where $S = (\mathbf{P}, D_S)$ is an inhabited *structure* and I is an *interpretation*, as follows.

The notion of structure generalizes the traditional notion of set. A structure $S = (\mathbf{P}, D_S)$ consists of a partially ordered set $\mathbf{P} = (P, \leq)$—P is called the set of nodes—and a functor D_S from $\mathbf{P}$ to the category of sets. For an inhabited structure, the sets must be nonempty. The structure S must be inhabited for the same reason that models of first-order logic must be inhabited: it allows for simpler rules for the quantifiers. So for each $\alpha \in P$ we have a nonempty set $D_S\alpha$; for $\alpha \leq \beta$ a function $(\sigma_S)^\alpha_\beta : D_S\alpha \to D_S\beta$; and for $\alpha \leq \beta \leq \gamma$, $(\sigma_S)^\alpha_\gamma = (\sigma_S)^\beta_\gamma(\sigma_S)^\alpha_\beta$. In this way an element does not have to exist above all nodes. For each number $n \geq 0$, let $S^n = (\mathbf{P}, D_{S^n})$ be the structure on $\mathbf{P}$ defined by $(D_{S^n})\alpha = (D_S\alpha)^n$ and by $(\sigma_{S^n})^\alpha_\beta = ((\sigma_S)^\alpha_\beta)^n$. For all α $(D_{S^0})\alpha$ is a singleton. A substructure of S is a structure $R = (\mathbf{P}, D_R)$ on $\mathbf{P}$ such that $D_R\alpha \subseteq D_S\alpha$ for all nodes α and such that its maps $(\sigma_R)^\alpha_\beta$ are restrictions of the maps $(\sigma_S)^\alpha_\beta$. A map $F : S \to T$ between structures $S = (\mathbf{P}, D_S)$ and $T = (\mathbf{P}, D_T)$ consists of a collection of functions $\{F_\alpha : D_S\alpha \to D_T\alpha \mid \alpha \in \mathbf{P}\}$, such that for all $\alpha \leq \beta$ the following diagram commutes:

$$\begin{tikzcd} D_S\alpha \arrow[r, "F_\alpha"] \arrow[d, "(\sigma_S)^\alpha_\beta"'] & D_T\alpha \arrow[d, "(\sigma_T)^\alpha_\beta"] \\ D_S\beta \arrow[r, "F_\beta"'] & D_T\beta \end{tikzcd}$$

The structures and maps above form the functor category $\mathbf{S}^{\mathbf{P}}$, where $\mathbf{S}$ is the category of sets.

The interpretation I in the definition of Kripke model assigns to each n-ary atomic predicate P of the language a substructure R of S^n and to each n-ary function symbol f a map $F : S^n \to S$, where constant symbols are interpreted as 0-ary functions. The equality predicate is interpreted by the equality relation $E = (\mathbf{P}, D_E)$ on S^2 defined by $D_E\alpha = \{(e, e) \mid e \in S\alpha\}$. A term t is interpreted as the composition T of the interpretations of its parts.

Given a Kripke model $\mathbf{K} = \langle(\mathbf{P}, D), I\rangle$, let $L_\mathbf{K}$ be the extension of the first-order language L obtained by including constant symbols for the elements of $\amalg_{\alpha\in P} D\alpha$. Write $\alpha \Vdash_{QS4} P(t, u, ...)$, in words, $P(t, u, ...)$ is *satisfied above* α, if $(T_\alpha, U_\alpha, ...) \in R\alpha$. Extend the satisfaction relation $\Vdash_{QS4}$ inductively to all sentences of $L_\mathbf{K}$ as follows: expression $\alpha \Vdash_{QS4} \varphi$ is *well–formed* if all new constant symbols of $L_\mathbf{K}$ occurring in φ are from $D\alpha$. In that case we sometimes write φ_α instead of φ. Given $\alpha \leq \beta$ and φ_α, φ_β is the formula constructed from φ_α by replacing the constant symbols for $c \in D\alpha$ by the constant symbols for $\sigma^\alpha_\beta(c) \in D\beta$. Use the same notation for constant symbols and constants, as it is clear from context which interpretation is intended. Extend the definition of $\Vdash_{QS4}$ inductively by:

$$\begin{aligned}
&\alpha \Vdash_{QS4} \top; \\
\alpha \Vdash_{QS4} (\varphi \wedge \psi)_\alpha &\iff \alpha \Vdash_{QS4} \varphi_\alpha \text{ and } \alpha \Vdash_{QS4} \psi_\alpha; \\
\alpha \Vdash_{QS4} (\varphi \vee \psi)_\alpha &\iff \alpha \Vdash_{QS4} \varphi_\alpha \text{ or } \alpha \Vdash_{QS4} \psi_\alpha; \\
\alpha \Vdash_{QS4} (\varphi \to \psi)_\alpha &\iff \alpha \Vdash_{QS4} \varphi_\alpha \text{ implies } \alpha \Vdash_{QS4} \psi_\alpha; \\
\alpha \Vdash_{QS4} (\neg\varphi)_\alpha &\iff \text{ it is not the case that } \alpha \Vdash_{QS4} \varphi_\alpha; \\
\alpha \Vdash_{QS4} (\forall x \varphi(x))_\alpha &\iff \alpha \Vdash_{QS4} \varphi(c)_\alpha \text{ for all } c \in D\alpha; \\
\alpha \Vdash_{QS4} (\exists x \varphi(x))_\alpha &\iff \alpha \Vdash_{QS4} \varphi(c)_\alpha \text{ for some } c \in D\alpha; \quad \text{and} \\
\alpha \Vdash_{QS4} (\Box\varphi)_\alpha &\iff \beta \Vdash_{QS4} \varphi_\beta \text{ for all } \beta \geq \alpha.
\end{aligned}$$

We write $\mathbf{K} \models_{QS4} \varphi$ if $\alpha \Vdash_{QS4} \varphi_\alpha$ for all nodes $\alpha \in P$. The completeness theorem for $QS4$ then reads: for all sets of sentences $\Gamma \cup \{\varphi\}$,

$$QS4, \Gamma \vdash \varphi \iff \text{ for all } \mathbf{K}, \text{ if } \mathbf{K} \models_{QS4} \gamma \text{ for all } \gamma \in \Gamma, \text{ then } \mathbf{K} \models_{QS4} \varphi.$$

Using the translation from intuitionistic predicate logic to $QS4$, we get the following interpretation for intuitionistic logic: Kripke models are pairs $\mathbf{K} = \langle S, I \rangle$ as before, with S an inhabited structure, and I assigning substructures, maps, and the equality relation to atomic predicates, function symbols, and the equality predicate. Define $\Vdash$ on atoms in exactly the same way as $\Vdash_{QS4}$. The extension of $\Vdash$ to the extended language of first-order intuitionistic predicate logic $L_{\mathbf{K}}$ differs, however: extend $\Vdash$ inductively by:

$$\begin{aligned}
&\alpha \Vdash \top; \\
\alpha \Vdash (\varphi \wedge \psi)_\alpha &\iff \alpha \Vdash \varphi_\alpha \text{ and } \alpha \Vdash \psi_\alpha; \\
\alpha \Vdash (\varphi \vee \psi)_\alpha &\iff \alpha \Vdash \varphi_\alpha \text{ or } \alpha \Vdash \psi_\alpha; \\
\alpha \Vdash (\varphi \rightarrow \psi)_\alpha &\iff \beta \Vdash \varphi_\beta \text{ implies } \beta \Vdash \psi_\beta, \text{ for all } \beta \geq \alpha; \\
\alpha \Vdash (\forall x \varphi(x))_\alpha &\iff \beta \Vdash \varphi(c)_\beta \text{ for all } \beta \geq \alpha \text{ and all } c \in D\beta; \quad \text{and} \\
\alpha \Vdash (\exists x \varphi(x))_\alpha &\iff \alpha \Vdash \varphi(c)_\alpha \text{ for some } c \in D\alpha.
\end{aligned}$$

For predication, equality, negation, and bi-implication this means:

$$\begin{aligned}
\alpha \Vdash P(t, u, ...)_\alpha &\iff (T_\alpha, U_\alpha, ...) \in R_\alpha; \\
\alpha \Vdash (t = u)_\alpha &\iff T_\alpha = U_\alpha; \\
\alpha \Vdash (\neg \varphi)_\alpha &\iff \text{it is not the case that } \beta \Vdash \varphi_\beta \text{ for any } \beta \geq \alpha; \quad \text{and} \\
\alpha \Vdash (\varphi \leftrightarrow \psi)_\alpha &\iff \text{for all } \beta \geq \alpha, \ \beta \Vdash \varphi_\beta \text{ if and only if } \beta \Vdash \psi_\beta.
\end{aligned}$$

We can easily verify that if $\alpha \leq \beta$ and $\alpha \Vdash \varphi_\alpha$, then $\beta \Vdash \varphi_\beta$.

Define $\mathbf{K} \models \varphi$ to mean that $\alpha \Vdash \varphi$ for all nodes $\alpha \in P$. The corresponding completeness theorem is: for all sets of sentences $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash \varphi \iff \text{ for all } \mathbf{K}, \text{ if } \mathbf{K} \models \gamma \text{ for all } \gamma \in \Gamma, \text{ then } \mathbf{K} \models \varphi.$$

A major example of the use of Kripke models occurs in forcing as developed by P. J. Cohen in 1963 ([Cohen 1966]). A generalized version is in [Keisler 1973] and is based on work in [Rasiowa, Sikorski 1963] on Boolean valued model theory. The generalization includes the forcing methods introduced by A. Robinson and J. Barwise. One stage consists of constructing a so-called forcing property: given a countable language L with countably many constant symbols, construct a partially ordered set $\mathbf{P}$ where the nodes are pairs $\mathbf{p} = (p, f_p)$ such that the f_p are finite sets of atomic sentences from L with $f_p \subseteq f_q$ whenever $(p, f_p) \leq (q, f_q)$. Construct a Kripke model $\mathbf{K}$ on $\mathbf{P}$ by assigning sets $D\mathbf{p} = \{c \mid \varphi c \in f_p \text{ for some } \varphi\}$ to $\mathbf{p} \in \mathbf{P}$, and relations R for each predicate n–ary P such that $R\mathbf{p} = \{\mathbf{c} \mid P\mathbf{c} \in f_p\}$, where $\mathbf{c} = (c_1, ..., c_n)$. The equality symbol is interpreted as a binary relation $\approx$ and need not be the standard equality relation of $\mathbf{K}$. Then $\mathbf{K}$ constitutes a *forcing property* if it satisfies

$$\mathbf{K} \models \forall x (\neg\neg x \approx x)$$

and

$$\mathbf{K} \models \forall xy ((x \approx y \wedge \varphi(x)) \rightarrow \neg\neg\varphi(y))$$

for all formulas φ where y is not bound by a quantifier of φ.

Given a forcing property $\mathbf{K}$, the forcing relation $\Vdash$ is defined as for intuitionistic forcing, except that the equality symbol $=$ is interpreted by $\approx$. Weak forcing $\Vdash_w$ is defined by $\mathbf{p}\Vdash_w\varphi$ if and only if $\mathbf{p}\Vdash\varphi^\bullet$, where $\varphi^\bullet$ is the double negation translation of φ. Forcing properties and the construction of generic models can now be considered as part of topos theory. The connection between topoi and independence proofs for the continuum hypothesis, was made by M. Tierney [Tierney 1972]. This was one of the main driving forces behind F. W. Lawvere and Tierney's development of topos theory.

§6. Topoi

Topos theory has its origins in three separate lines of mathematical development. In the introduction of [Johnstone 1977] we find the first and the third: sheaf theory and the category-theoretic foundation of mathematics. The second line is expounded in the preface of [Goldblatt 1979].

The first line is sheaf theory, which was developed after the second world war as a tool for algebraic topology. Later the concept of a sheaf over a topological space was extended to that of a sheaf over a site in order to enable the construction of more "topologies" in algebraic geometry. Categories of sheaves over a site are known as Grothendieck topoi.

The second line of development was initiated by Cohen's forcing technique in his proof of the independence of the continuum hypothesis. It was realized very early that Cohen's forcing and Kripke's forcing were closely related examples of some common technique connecting intuitionistic logic and classical set theory. The topic was picked up by D. S. Scott and R. Solovay in 1965, when they developed the theory of Boolean-valued models of ZF. In the late 1960's Scott considered the natural generalization to Heyting-valued models.

The third line involved the attempts to axiomatize category-theoretically well-known categories such as module categories. P. T. Johnstone [Johnstone 1977] mentioned as an early example the proof of the Lubkin–Heron–Freyd–Mitchell embedding theorem [Freyd 1964] for abelian categories, which showed that there is an explicit set of elementary axioms that imply all the finitary exactness properties of module categories. Lawvere tried the same for the category of sets.

Cartesian closed categories preceded topoi. In fact, a topos is a finitely complete Cartesian closed category satisfying one extra property. A category is finitely complete if all finite diagrams have limits and colimits. A finitely complete Cartesian closed category $\mathbf{C}$ has all finite limits and colimits explicitly given by functors, and the functors $- \times b : \mathbf{C} \to \mathbf{C}$ have right adjoints. The category theoretic axioms of forming new arrows from previously constructed ones are represented as sequent rules and axioms below. The letters A, B, C, $\cdots$ represent object variables, and f, g, h, $\cdots$ are arrow variables. The arrows 0_A, 1_A and id_A are introduced as axioms. A thin horizontal line means that if the arrows above the line exist, then there exists a corresponding arrow below the line. A fat line means the same as a thin line except that the correspondence is one–to–one and onto.

$$A \xrightarrow{id_A} A$$

$$\frac{A \xrightarrow{f} B \quad B \xrightarrow{g} C}{A \xrightarrow{gf} C}$$

$$A \xrightarrow{1_A} 1 \qquad\qquad 0 \xrightarrow{0_A} A$$

$$\frac{A \xrightarrow{f} B \quad A \xrightarrow{g} C}{A \xrightarrow{<f,g>} B \times C} \qquad \frac{B \xrightarrow{f} A \quad C \xrightarrow{g} A}{B \amalg C \xrightarrow{f+g} A}$$

$$\frac{A \times B \xrightarrow{f} C}{A \xrightarrow{\varphi f} C^B}$$

The notion of finitely complete Cartesian closed category is a straightforward generalization of intuitionistic propositional logic: take the formulas of a propositional logic as objects and have a unique arrow from φ to ψ exactly when $\varphi \vdash \psi$. Thus models of intuitionistic propositional logic give rise to examples of finitely complete Cartesian closed categories. This also implies that the notion of Cartesian closed category is too weak to capture a sufficient number of properties of higher order set theory.

Grothendieck topoi, on the other hand, do reflect a substantial part of set theory. A more detailed description of and references to the development of the notion of Grothendieck topos can be found in [Gray 1979]. Of main interest to us is Giraud's Theorem (1963–1964), which characterizes Grothendieck topoi by certain exactness conditions ([Johnstone 1977], pp. 15–18). Lawvere considered Grothendieck topoi as models for his generalized set theory. His early axiomatization, however, still included some set-theoretic aspects, and was therefore not purely category-theoretical. An adaptation of Giraud's exactness conditions could provide a generalized set theory axiomatized purely in terms of finite exactness conditions and constructions. Lawvere's discovery that each Grothendieck topos has a subobject classifier $t : 1 \to \Omega$ completed that picture. It then turned out that a small number of finite exactness conditions, plus the existence of a subobject classifier, sufficed to develop a category-theoretic notion of set theory. During the year 1969–1970 in Halifax Lawvere and Tierney developed the fundamentals of elementary topos theory.

An elementary *topos* is a finitely complete Cartesian closed category **E** that has a subobject classifier. The existence of finite colimits follows from the other axioms ([Mikkelsen 1976], [Paré 1974]), so a topos **E** only has to satisfy:

(1) **E** has all finite products and equalizers.
(2) **E** is Cartesian closed: The functor $b \mapsto b \times a$ has a right adjoint functor $b \mapsto b^a$, for all a.
(3) **E** has a *subobject classifier* $t : 1 \to \Omega$. That is, for each monomorphism $f : a \to b$ there is a unique χ_f, the *classifying map*, from b to the *truth value object* Ω such that the following is a pullback:

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \chi_f} \\ 1 & \xrightarrow{t} & \Omega \end{array}$$

In the definition above we assume that all limits are given functorially. For (1), finite products and equalizers are sufficient to construct all finite limits [Mac Lane 1971, p. 109]. For (2), the categories b^a behave like the function sets b^a in the category $\mathbf{S}$ of sets. The map χ_f of (3) then behaves like the characteristic function of the image of f. In fact, the category of sets $\mathbf{S}$ is a topos, with $\Omega \cong \{0, 1\}$.

Topoi have an internal logical structure just like $\mathbf{S}$: intuitionistic type theory. This was first made explicit by W. Mitchell [Mitchell 1972]. A topos need not satisfy any additional choice principles, like Dependent Choice or Countable Choice. It does however have full comprehension for each object, and a power–set construction: for each A, Ω^A is its power–object. Both category theory and intuitionism worked in a field that in some way extended the traditional universe of classical logic and sets. That both fields meet in topos theory suggests that they satisfy S. Mac Lane's dictum: good general theory does not search for the maximum generality, but for the right generality [Mac Lane 1971, p. 103].

Colimits are derivable from the definition above for the same reason that existential quantification and disjunction (and, in fact, conjunction and negation) are definable in terms of universal quantification and implication over Ω ([Prawitz 1965], [Scott 1979]):

$$\begin{aligned} \vdash (p \wedge q) &\leftrightarrow \forall r((p \to (q \to r)) \to r) \\ \vdash (p \vee q) &\leftrightarrow \forall r(((p \to r) \wedge (q \to r)) \to r) \\ \vdash (\neg p) &\leftrightarrow \forall r(p \to r) \\ \vdash (\exists x \varphi(x)) &\leftrightarrow \forall r(\forall x(\varphi(x) \to r) \to r), \end{aligned}$$

where p, q and r are variables over the set of truth values Ω. Similarly, the union of two subsets X and Y of a set S is the intersection of all the subsets of S containing both X and Y.

Let $\mathbf{C}$ be a small category. Then the functor category $\mathbf{S}^{\mathbf{C}}$ is a topos. Objects of $\mathbf{S}^{\mathbf{C}}$ are also called *presheaves*. So Kripke models are presheaves. Finite limits and colimits are created pointwise: $(X \times Y)(a) = X(a) \times Y(a)$ for all a; $1(a) = \{0\}$ for all a; and if F is the coequalizer of $f, g : X \to Y$, then for all a $F(a)$ is the coequalizer of $f(a)$ and $g(a)$. Let X and Y be presheaves of $\mathbf{S}^{\mathbf{C}}$. For the exponent presheaf Y^X we consider "local" natural transformations. Just taking function sets $Y^X(a) = Y(a)^{X(a)}$ usually does not work: functions at node a must be provided with information as to what they look like at later stages x, reached from a by maps $f : a \to x$. Let a be an object of $\mathbf{C}$. The comma category $\mathbf{C} \uparrow a$ has as objects arrows $f : a \to x$ and $g : a \to y$, and as arrows maps $w : f \to g$, with $w : x \to y$ from $\mathbf{C}$, satisfying $wf = g$. There are induced functors X_a and Y_a from $\mathbf{C} \uparrow a$ to $\mathbf{S}$ defined by $X_a(f) = X(x)$, $X_a(w) = X(w)$, $Y_a(f) = Y(x)$ and $Y_a(w) = Y(w)$. Set $Y^X(a)$ equal to the set of natural transformations from X_a to Y_a. For $p : b \to a$, define $Y^X(p) : Y^X(b) \to Y^X(a)$ by $Y^X(\rho)_w = \rho_{wp}$. As to the truth value object Ω, let a be an object of $\mathbf{C}$. Set $\Omega(a)$ to be the set of subobjects $\mathbf{s}$ in $\mathbf{S}^{\mathbf{C}}$ of the representable functor $\mathbf{C}(a, -) : \mathbf{C} \to \mathbf{S}$. So $\mathbf{s}(x) \subseteq \mathbf{C}(a, x)$ for all objects and arrows x of $\mathbf{C}$. For $p : a \to b$, define $\Omega(p) : \Omega(a) \to \Omega(b)$ by $\Omega(p)(\mathbf{s}) = \mathbf{t}$ with $f \in \mathbf{t}$ if and only if $fp \in \mathbf{s}$.

It is possible to construct subtopoi of *sheaves* $\mathrm{sh}_j(\mathbf{E})$ of a topos $\mathbf{E}$ by restricting the number of possible truth values of Ω using a topology j, not to be confused with, and quite different from, a topology on a set. A Grothendieck topos is a category equivalent to a topos of the form $\mathrm{sh}_j(\mathbf{S}^{\mathbf{C}})$, so Grothendieck topoi are elementary topoi. All models of first-order intuitionistic logic mentioned in §5 are sheaves in Grothendieck topoi.

The definition of topology went through several generalizations until Lawvere arrived at the following definition: a *topology* on a topos $\mathbf{E}$ is a map $j : \Omega \to \Omega$

satisfying (1) $j \cdot t = t$; (2) $j \cdot j = j$; and (3) $j \cdot \wedge = \wedge \cdot (j \times j)$, where $\wedge$ is the intersection map, the classifying map of $\langle t, t\rangle : 1 \to \Omega \times \Omega$. The subobject Ω_j of Ω of fixed points of j represents the remaining possible truth values and is the truth value object in $\mathrm{sh}_j(\mathbf{E})$.

Rather than describe the general procedure of making a Grothendieck topos, we illustrate the construction for the special case of sheaves over a topological space. Let X be a topological space, and let $\mathbf{S}^{\mathbf{P}}$ be the topos of presheaves on the partially ordered set $\mathbf{P} = O(X)^{op}$. The set $O(X)$ is the partially ordered set of open subsets of X ordered by inclusion. The dual category $\mathbf{P}$ is therefore the lattice of open subsets of X ordered by containment $U \leq V$ if and only if $U \supseteq V$. Presheaves S consist of sets $S(U)$ for all $U \in O(X)$ and restriction maps $\sigma^U_V : S(U) \to S(V)$ for all pairs of open sets $U \supseteq V$. The truth value object Ω is defined by $\Omega(U) = \{S \subseteq O(U) \mid W \subseteq V \in S \text{ implies } W \in S\}$. Define a topos-theoretic topology j that relates to the set-theoretic topology $O(X)$ by setting maps $j_U : \Omega(U) \to \Omega(U)$ such that $j_U(S) = \{V \in O(U) \mid V \subseteq \cup S\}$. Then Ω_j is such that $\Omega_j(U) = O(U)$ and $(\Omega_j)^U_V(W) = V \cap W$. The resulting topos $\mathrm{sh}_j(\mathbf{S}^{\mathbf{P}}) = \mathrm{sh}(X)$ is the subcategory of sheaves of $\mathbf{S}^{\mathbf{P}}$, that is, the presheaves R satisfying

(1) If $S \subseteq O(U)$ is such that $\cup S = U$, and $x, y \in R(U)$ are such that $\sigma^U_V(x) = \sigma^U_V(y)$ for all $V \in S$, then $x = y$.
(2) If $S \subseteq O(U)$ is such that $\cup S = U$, and there are elements $x_V \in R(V)$ such that $\sigma^V_{V\cap W}(x_V) = \sigma^W_{V\cap W}(x_W)$ for all $V, W \in S$, then there exists $x \in R(U)$ such that $\sigma^U_V(x) = x_V$ for all $V \in S$.

Grothendieck topoi also have natural number objects. In functor categories $\mathbf{S}^{\mathbf{P}}$ the constant presheaf $\mathbf{N}$ defined by $\mathbf{N}(a) = \omega$ for all a performs this role. Lawvere's original definition of natural number object is equivalent to P. Freyd's elementary characterization [Freyd 1972]: A topos has a *natural number object* $\mathbf{N}$ if there exist arrows $o : 1 \to \mathbf{N}$ and $s : \mathbf{N} \to \mathbf{N}$ (zero and successor) such that $1 \xrightarrow{o} \mathbf{N} \xleftarrow{s} \mathbf{N}$ is a coproduct diagram and $\mathbf{N} \to 1$ is a coequalizer of s and $id_N : \mathbf{N} \to \mathbf{N}$. In Grothendieck topoi the natural number object satisfies all first-order statements of classical number theory, but in the general situation of elementary topoi with natural number object it only has to satisfy the higher order equivalent of HA.

Cohen's forcing and many methods used in independence proofs of set theory are in fact topos theoretic techniques ([Tierney 1972], [Ščedrov 1984]). There are many more applications of internal intuitionistic logic via topos theory. Proceedings like [Fourman et al. 1979] and [Troelstra, van Dalen 1982], and monographs like [Kock 1981] and [Ščedrov 1984] show just a few examples of the possible applications of intuitionistic logic and topos theory to such areas as Banach spaces, analysis, sheaf theory, topology, differential geometry, complex variables, algebra and set theory. The applications to classical mathematics confirm in a concrete way that proving something constructively really means proving something more.

Brouwer provided classical proofs for his results in topology and later denounced these proofs as being insufficient for an intuitionist. It will be ironic if intuitionistic methods and sheaf models provide a method to prove Brouwer's contributions to topology.

It was the discovery of the effective topos *Eff* by M. Hyland and the subsequent development of tripos theory that gave rise to topoi, which bring into topos theory the previous models of realizability and the Dialectica interpretation ([Hyland 1982], [Hyland et al. 1980]). In an unpublished manuscript in 1977 W. Powell formulated in a classical context a semantics for realizability that has analogies with

Hyland's approach. A significant difference between old style realizability and *Eff*-style topoi is that realizability gives interpretations of logical structures, while *Eff* allows us to think of realizability as model theory. The truth value of a sentence A in *Eff* is the set of numbers e such that $e\mathbf{r}A$.

Currently topos theory is the unifying concept behind the unintended interpretations of intuitionistic logic.

§7. Intended Interpretations of Intuitionistic Logic

With so many unintended interpretations available for intuitionistic logic, we might at first expect that there should be at least one undisputed proper interpretation. That this is not the case can be explained as follows: an unintended interpretation presents no reason for dispute since it is, after all, unintended. An intended interpretation has to reflect how a *real* intuitionist—Brouwer, say—interprets the connectives. Since Brouwer never made an attempt at this himself, we seem to get closest to it by considering Heyting's proof interpretation for first-order logic and its extensions by G. Kreisel.

The following describes the Brouwer–Heyting–Kreisel (BHK) proof interpretation. A statement φ is true only if we have a proof p for it which satisfies the following requirements:

(1) p proves $\varphi \wedge \psi$ just in case p consists of a pair q, r of proofs of φ and ψ, respectively.
(2) p proves $\varphi \vee \psi$ just in case p consists of a pair n, q such that either $n = 0$ and q proves φ or $n = 1$ and q proves ψ.
(3) p proves $\varphi \to \psi$ just in case p consists of a pair q, r such that q is a construction that converts each proof s of φ into a proof $q(s)$ of ψ and such that r is a proof that q is such a construction.
(4) p proves $\exists x A(x)$ just in case p consists of a pair q, r such that q is a construction that yields an element c such that r is a proof of $A(c)$.
(5) p proves $\forall x A(x)$ just in case p consists of a pair q, r such that for all c in the domain, $q(c)$ is a proof of $A(c)$, and such that r is a proof that q is so.

Kreisel [Kreisel 1962] proposed the addition of the "extra proof r" clauses in the descriptions of the $\to$-case and the $\forall$-case. The interpretation is not reductive: it does not break proofs p down into simpler notions. Parts of the proof interpretation have been brought into question (for references, see [van Dalen 1982, p. 61]). Questions arose as to whether one can quantify over a universe of *all* proofs, or whether the extra proof r is of a nature similar to proofs p. Intuitionists see the proof interpretation as an *explanation* rather than an interpretation of intuitionistic logic. Moreover, as Brouwer indicated, (formal) language is not trustworthy, and from that point of view the dispute is not surprising. Heyting's formalization itself, however, appears to be undisputed.

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