Basic Propositional Calculus I

Mohammad Ardeshir
Wim Ruitenburg

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Department of Mathematics, Statistics and Computer Science
Marquette University
P.O. Box 1881
Milwaukee, WI 53201 USA

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Mohammad Ardeshir[1]
Department of Mathematics
Sharif University of Technology
P.O. Box 11365-9415
Tehran
Iran
ardeshir@math.sharif.ac.ir
Wim Ruitenburg[2]
Department of Mathematics, Statistics and Computer Science
Marquette University
P.O. Box 1881
Milwaukee, WI 53201
wimr@mscs.mu.edu

Abstract

We present an axiomatization for Basic Propositional Calculus BPC, and give a completeness theorem for the class of transitive Kripke structures. We present several refinements, including a completeness theorem for irreflexive trees. The class of intermediate logics includes two maximal nodes, one being Classical Propositional Calculus CPC, the other being E_1, a theory axiomatized by $\top \to \bot$. The intersection $\mathrm{CPC} \cap E_1$ is axiomatizable by the Principle of the Excluded Middle $A \vee \neg A$. If B is a formula such that $(\top \to B) \to B$ is not derivable, then the lattice of formulas built from one propositional variable p using only the binary connectives, is isomorphically preserved if B is substituted for p. A formula $(\top \to B) \to B$ is derivable, exactly when B is provably equivalent to a formula of the form $((\top \to A) \to A) \to (\top \to A)$.

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1 Introduction

There exist two principal sources of justification for Basic Logic; constructivism and constructivity. The one through constructivism is somewhat more philosophical, and expounded in [7]. In that paper we observe that the first-order logic of the well-known constructivisms is Intuitionistic Predicate Calculus IQC. Most explanations of the logical constants are variations upon the Brouwer-Heyting-Kolmogorov proof interpretation. Our modification of this interpretation yields the first-order logic Basic Predicate Calculus BQC, a proper subsystem of IQC.

The approach through constructivity precedes the one through constructivism, but was originally restricted to Basic Propositional Calculus BPC, see [10]. The traditional world of constructivity consisted of two related, but quite distinct, parts. Constructivity considers the tools and machinery that are associated with or motivated by constructivism or by computability. So one part includes the forms of constructive mathematics alluded to above; the other involves general recursive function

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theory. In recursive 'constructive' mathematics one often considers situations where all proofs, in most cases codes for programs that correspond with proofs, are essentially known. Not all such situations, like in the case of the encodings used in the well-known proof of Gödel's incompleteness theorem, are traditionally recognized as a form of constructivity. If we accept the more inclusive notion as constructivity, then this proof of Gödel's incompleteness theorem provides us with another constructive logic, the modal logic for provability PrL (identical to the system G of [3]), which satisfies Löb's rule. The syntax of Intuitionistic Propositional Calculus IPC differs in an essential way from that of PrL with its $\Box$, but IPC corresponds in a natural way to the modal logic S4. Albert Visser found a propositional logic, FPC, with a syntax identical to the one for IPC, to complete the informal equation

$$\frac{\mathrm{IPC}}{\mathrm{S4}} = \frac{\mathrm{FPC}}{\mathrm{PrL}}.$$

This enabled him to interpret implication as formal provability [10]. In [9] Craig Smoryński uses Basic Modal Logic BML, better known as K4, as the natural generalization of both S4 and PrL. Similarly, Visser went beyond the equation above by introducing the system BPC satisfying the informal equations

$$\frac{\mathrm{BPC}}{\mathrm{K4}} = \frac{\mathrm{IPC}}{\mathrm{S4}} = \frac{\mathrm{FPC}}{\mathrm{PrL}}.$$

In this paper we present the first part of an overview of BPC and its model theory. Many results for Basic Predicate Calculus BQC and its model theory are natural generalizations of results known for Intuitionistic Propositional Calculus IPC. Therefore this study of BPC precedes the model theory of BQC and further extensions. We discuss the fundamental algebraic aspects of BPC; Kripke model theory for BPC, which appears to offer the easiest methods to derive additional results on intermediate logics.

2 BPC Axioms and Rules

The language for Basic Propositional Calculus BPC is the same as the one for Intuitionistic Propositional Calculus IPC. So it has a set of propositional variables, the usual logical constants $\top$ and $\bot$, and the logical connectives $\wedge$, $\vee$, and $\rightarrow$. The theory of BPC is a proper subsystem of IPC, but with restriction to $\top$, $\bot$, $\wedge$, and $\vee$ it still is the usual one that, as for IPC, corresponds to a distributive lattice with top and bottom. The fraction that involves $\rightarrow$ is substantially weaker. It is no longer true that, for all A, the map $Y \mapsto A \rightarrow Y$ functions as right adjoint to the map $X \mapsto X \wedge A$. In particular, modus ponens doesn't hold in BPC.

There exist several ways to axiomatize BPC. In [10] we encounter a version that uses natural deduction. In [1] further axiomatizations are considered, including cut free versions. In [6] and [7], and here, we use a version with sequents. The BPC axiomatization below is given in the form of a collection of sequent axioms and axiom rules. We use notational conventions as illustrated below. For the rules a single horizontal line means that if the sequents above the line hold, then so do the ones below the line. A double line means the same, but in both directions. The BPC axioms that don't involve $\rightarrow$ are essentially those for a distributive lattice with top and bottom. So BPC satisfies all substitution instances of:

$$A \Rightarrow A$$

$$\frac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}$$

$$A \Rightarrow \top \qquad\qquad \bot \Rightarrow A$$

$$\frac{A \Rightarrow B \quad A \Rightarrow C}{A \Rightarrow B \wedge C} \qquad \frac{B \Rightarrow A \quad C \Rightarrow A}{B \vee C \Rightarrow A}$$

$$A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$$

The implication-free fragment is identical to the proposition logical fragment of geometric logic. The system BPC diverges from Intuitionistic Propositional Calculus IPC in its rules and axioms for implication. BPC satisfies all substitution instances of

$$\frac{A \wedge B \Rightarrow C}{A \Rightarrow B \to C}$$

Notice the single horizontal line as opposed to the double line in case of IPC. We also add the 'formalized' versions of some of the rules of $\Rightarrow$ to make $\to$ reflect its properties, see Proposition 2.3:

$$(A \to B) \wedge (B \to C) \Rightarrow A \to C$$

$$(A \to B) \wedge (A \to C) \Rightarrow A \to (B \wedge C)$$

$$(B \to A) \wedge (C \to A) \Rightarrow (B \vee C) \to A$$

This completes the axiomatization of BPC.

The expressions $\neg A$ and $A \leftrightarrow B$ are the usual abbreviations for $A \to \bot$ and $(A \to B) \wedge (B \to A)$, respectively. We write $A \Leftrightarrow B$ as short for $A \Rightarrow B$ plus $B \Rightarrow A$, and often $\Rightarrow A$, or even A, for $\top \Rightarrow A$. Given a set Γ of sequents and rules, let $\mathrm{Cl}(\Gamma)$ be the set of all sequents that can be obtained, after finitely many applications of the BPC rules and the rules of Γ, from the BPC sequent axioms plus the axiom sequents of Γ. We say Γ *satisfies*, or *proves*, $A \Rightarrow B$, written $\Gamma \vdash A \Rightarrow B$, when $A \Rightarrow B \in \mathrm{Cl}(\Gamma)$. Similarly, Γ *satisfies*, or *proves*, the rule

$$\gamma = \frac{A_1 \Rightarrow B_1 \quad \ldots \quad A_n \Rightarrow B_n}{A \Rightarrow B},$$

written $\Gamma \vdash \gamma$, when $\Gamma \cup \{A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n\} \vdash (A \Rightarrow B)$. We usually write $\Gamma \vdash A$ as short for $\Gamma \vdash\Rightarrow A$. The set Γ is *consistent* when $\Gamma \nvdash \bot$. A *theory* Γ over BPC is a set of sequents and rules closed under derivability. A theory is *axiomatizable* by a set Γ if it equals the closure of Γ under derivability. A theory Γ is a *sequent theory* if it is axiomatizable by a set of sequents. In that case Γ is uniquely determined by $\mathrm{Cl}(\Gamma)$. For that reason we identify sequent theories with sets $\mathrm{Cl}(\Gamma)$ of sequents closed under derivability.

The standard examples of theories are IPC, FPC, and CPC: Intuitionistic Propositional Calculus IPC is the extension of BPC by all substitution instances of the Rule of Modus Ponens

$$\frac{A \Rightarrow B \to C}{A \wedge B \Rightarrow C}.$$

Formal Propositional Calculus FPC is the extension of BPC by all substitution instances of Löb's Rule

$$\frac{A \wedge (\top \to B) \Rightarrow B}{A \Rightarrow B}.$$

Classical Propositional Calculus CPC is the extension of IPC by all substitution instances of Excluded Middle

$$\Rightarrow A \vee \neg A.$$

Our first task is showing that IPC and FPC, and thus CPC, are sequent theories.

Proposition 2.1 *IPC is axiomatizable by the schema*

$$\top \to A \Rightarrow A.$$

FPC is axiomatizable by the schema (Löb's Axiom)

$$(\top \to A) \to A \Rightarrow \top \to A.$$

Proof. Obviously, the axiom schema $\top \to A \Rightarrow A$ is derivable from IPC. Conversely, we easily see that BPC satisfies the following weakening of Modus Ponens:

$$\frac{A \Rightarrow B \to C}{A \wedge B \Rightarrow \top \to C}.$$

This, combined with $\top \to C \Rightarrow C$, gives the Rule of Modus Ponens. Apply Löb's Rule to $((\top \to A) \to A) \wedge (\top \to (\top \to A)) \Rightarrow \top \to A$ to derive Löb's Axiom. Conversely, suppose $A \wedge (\top \to B) \Rightarrow B$. Then $A \Rightarrow (\top \to B) \to B$ hence, by Löb's Axiom, $A \Rightarrow \top \to B$. So $A \Rightarrow A \wedge (\top \to B)$, and thus $A \Rightarrow B$. ⊣

Note that FPC $\cup$ IPC is inconsistent.

Proposition 2.2 (Functional Completeness) *Let* Γ *be a sequent theory. Then*

$$\Gamma \cup \{A\} \vdash B \Rightarrow C$$

if and only if

$$\Gamma \vdash A \wedge B \Rightarrow C.$$

Proof. By induction on the complexity of proofs. If $\Gamma \vdash B \Rightarrow C$ then, obviously, $\Gamma \vdash A \wedge B \Rightarrow C$. The case where B equals $\top$ and C equals A is trivial. Suppose that $B \Rightarrow C$ follows from $B \Rightarrow D$ and $D \Rightarrow C$. By induction, $\Gamma \vdash (A \wedge B \Rightarrow D)$ and $\Gamma \vdash (A \wedge D \Rightarrow C)$. Then $\Gamma \vdash (A \wedge B \Rightarrow A \wedge D)$, and thus, by transitivity, $\Gamma \vdash (A \wedge B \Rightarrow C)$. Suppose B is of the form $D \vee E$, and $D \vee E \Rightarrow C$ follows from $D \Rightarrow C$ and $E \Rightarrow C$. By induction, $A \wedge D \Rightarrow C$ and $A \wedge E \Rightarrow C$ follow from Γ, so $(A \wedge D) \vee (A \wedge E) \Rightarrow C$ does too. Apply distributivity. The other cases are similar. The reverse trivially holds. ⊣

The axiomatization of BPC includes 'formalized' versions of some rules of the axiomatization: In the formalization one has, among other things, $\to$ in places where in the corresponding positions in the original rule one sees $\Rightarrow$. This formalization extends to all rules as follows:

Proposition 2.3 (Formalization) *Let* Γ *be a sequent theory. Then*

$$\Gamma \cup \{A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n\} \vdash (A \Rightarrow B)$$

implies

$$\Gamma \vdash (A_1 \to B_1) \wedge \ldots \wedge (A_n \to B_n) \Rightarrow A \to B.$$

Proof. By induction on the complexity of proofs. The rule of Implication Introduction immediately implies that if $\Gamma \vdash A \Rightarrow B$, then $\Gamma \vdash X \Rightarrow A \to B$, where X is the required conjunction of implications above. If $A \Rightarrow B$ equals $A_i \Rightarrow B_i$ for some i, then we even have $\vdash X \Rightarrow A \to B$. As to the rules, suppose that $A \Rightarrow B$ follows, by the Transitivity Rule, from $A \Rightarrow C$ and $C \Rightarrow B$. By induction, $\Gamma \vdash X \Rightarrow A \to C$ and $\Gamma \vdash X \Rightarrow C \to B$, so $\Gamma \vdash X \Rightarrow (A \to C) \wedge (C \to B)$. Application of the 'formalized' Transitivity Axiom, plus the Transitivity Rule, then yields $\Gamma \vdash X \Rightarrow A \to B$. Suppose that $A \Rightarrow B \to C$ follows, by Implication Introduction, from $A \wedge B \Rightarrow C$. By induction, $\Gamma \vdash X \Rightarrow A \wedge B \to C$. Now

$A \Rightarrow B \to A \wedge B$ holds, so by the axiom sequent of 'formalized' transitivity we have $A \wedge (A \wedge B \to C) \Rightarrow B \to C$. So $A \wedge B \to C \Rightarrow A \to (B \to C)$ by Implication Introduction, and thus $\Gamma \vdash X \Rightarrow A \to (B \to C)$. The remaining cases are just as simple. $\dashv$

This Proposition shows that implication $\to$ dutifully reflects the properties of the sequent arrow $\Rightarrow$. Implication $\to$ reflects $\Rightarrow$ properly if the converse also holds: A set of sequents Γ is called *faithful* if

$$\Gamma \vdash (A_1 \to B_1) \wedge \ldots \wedge (A_n \to B_n) \Rightarrow A \to B$$

implies

$$\Gamma \cup \{A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n\} \vdash (A \Rightarrow B).$$

All extensions of IPC are faithful since, in that case, $C \Rightarrow D$ is equivalent to $C \to D$. In particular CPC is faithful. Let E_1 be the theory axiomatized by $\top \to \bot$. Then $E_1 \vdash A \to B$, for all A and B, so it essentially says that we can replace all implications in the axioms and rules of BPC by $\top$. So we end up with geometric propositional logic, which is consistent. If E_1 were faithful, then $E_1 \vdash \bot$, so E_1 were inconsistent; contradiction. So E_1 is not faithful. We will show below that BPC and FPC are faithful.

Contexts are defined as follows: Add a new propositional variable, say P, to the language. Let D be a formula over the extended language, and let A be a formula over the old language. Then $D[A]$ is constructed by replacing each occurrence of P by A. Similarly, multiple, say double, substitution is performed by adding two new atoms P and Q to the language; D is a formula over the extended language; and A and B are formulas over the old language. Then $D[A, B]$ is formed by replacing all occurrences of P by A, and all occurrences of Q by B.

Proposition 2.4 (Substitution) *BPC satisfies the Substitution Rule*

$$\frac{A \wedge B \Rightarrow C \quad A \wedge C \Rightarrow B}{A \wedge D[B] \Rightarrow D[C]}$$

and the Substitution Axiom

$$A \wedge B \leftrightarrow A \wedge C \Rightarrow A \wedge D[B] \leftrightarrow A \wedge D[C].$$

Proof. By Proposition 2.3 the Substitution Rule for BPC immediately implies the Substitution Axiom, so it suffices to prove the Rule. Let Γ be the theory axiomatized by $A \wedge B \Rightarrow C$ and $A \wedge C \Rightarrow B$. We complete the proof by induction on the complexity of D. The cases for D a propositional variable or a constant are trivial. If $D[p]$ equals a disjunction $E[p] \vee F[p]$, then, by induction, we have $\Gamma \vdash A \wedge E[B] \Rightarrow E[C] \vee F[C]$ and $\Gamma \vdash A \wedge F[B] \Rightarrow E[C] \vee F[C]$. By the Disjunction Rule and Distributivity, $\Gamma \vdash A \wedge (E[B] \vee F[B]) \Rightarrow E[C] \vee F[C]$. If $D[p]$ equals a conjunction $E[p] \wedge F[p]$, then, by induction, we have $\Gamma \vdash (A \wedge F[B]) \wedge E[B] \Rightarrow E[C]$ and $\Gamma \vdash (A \wedge E[B]) \wedge F[B] \Rightarrow F[C]$. So $\Gamma \vdash A \wedge (E[B] \wedge F[B]) \Rightarrow E[C] \wedge F[C]$. Finally, suppose $D[p]$ equals an implication $E[p] \to F[p]$. By induction we have $\Gamma \vdash A \wedge E[C] \Rightarrow E[B]$ and $\Gamma \vdash A \wedge F[B] \Rightarrow F[C]$, so $\Gamma \vdash A \Rightarrow (E[C] \to E[B]) \wedge (F[B] \to F[C])$. Thus $\Gamma \vdash A \wedge (E[B] \to F[B]) \Rightarrow E[C] \to F[C]$. This completes the proof by induction. $\dashv$

A propositional variable P occurs *formally* in a formula $A[P]$ if P only occurs inside implication subformulas; P is *strictly informal* in $A[P]$ if P does not occur inside implication subformulas.

Proposition 2.5 (Formal Substitution) *Let P be formal in $A[P]$. Then BPC satisfies*

$$(B \leftrightarrow C) \wedge A[B] \Rightarrow A[C].$$

Proof. The result follows immediately from Substitution when $A[P]$ is of the form $D[P] \rightarrow E[P]$. The general case then follows by induction on the complexity of $A[P]$, identical to the proof of Substitution above. $\dashv$

Let Q be a propositional variable. Then the sets $\mathcal{P}$ and $\mathcal{N}$ are inductively defined by

$\{Q, A \wedge P, P \wedge A, A \vee P, P \vee A, A \rightarrow P, N \rightarrow A\} \subseteq \mathcal{P}$ and
$\{A \wedge N, N \wedge A, A \vee N, N \vee A, A \rightarrow N, P \rightarrow A\} \subseteq \mathcal{N}$,

for all A in which Q does not occur, $P \in \mathcal{P}$, and $N \in \mathcal{N}$.

Clearly, the collection of formulas in which Q occurs exactly once is the disjoint union of the collections $\mathcal{P}$ and $\mathcal{N}$ of *positive contexts* and *negative contexts* of Q. Each context $A[P]$ is essentially a repeated substitution of P in places marked by uniquely occurring propositional variables $Q_1, \ldots, Q_n$ in some formula B. A propositional variable P is *positive* (*negative*) in a formula $A[P]$ if each such substitution takes place in a variable Q_i for which B is a positive (negative) context. Note that if P is strictly informal in $A[P]$, then it is also positive in $A[P]$.

Proposition 2.6 (Monotonicity) *Let P be positive in $A[P]$. Then BPC satisfies*

$$\frac{B \wedge C \Rightarrow D}{B \wedge A[C] \Rightarrow A[D]}.$$

Let P be negative in $A[P]$, then BPC satisfies

$$\frac{B \wedge C \Rightarrow D}{B \wedge A[D] \Rightarrow A[C]}.$$

Proof. We may assume that P occurs exactly once in $A[P]$. We complete the proof of both statements simultaneously by induction on the complexity of $A[P]$. The cases where $A[P]$ is a propositional variable or constant are trivial. Suppose P is positive in $A[P]$, which equals $E[P] \rightarrow F[P]$. By induction, we derive from $B \wedge C \Rightarrow D$ both $B \wedge F[C] \Rightarrow F[D]$ and $B \wedge E[D] \Rightarrow E[C]$. So we also get $A \wedge (E[C] \rightarrow F[C]) \Rightarrow E[D] \rightarrow F[D]$. All other cases are just as easy. $\dashv$

In the same way that Proposition 2.5 is the 'formalized' companion of Substitution, the following is the 'formalized' companion of Monotonicity: If P is formal and positive in $A[P]$, then BPC satisfies

$$(C \rightarrow D) \wedge A[C] \Rightarrow A[D].$$

If P is formal and negative in $A[P]$, then BPC satisfies

$$(C \rightarrow D) \wedge A[D] \Rightarrow A[C].$$

Lemma 2.7 *BPC satisfies $A[A[\top]] \wedge (\top \rightarrow A[\top]) \Leftrightarrow A[\top]$.*

Proof. The direction $\Leftarrow$ easily follows with Substitution. For the converse: We can write $A[P]$ as $B[P, P]$, where Q occurs formally and R occurs strictly informally in $B[Q, R]$. Then BPC satisfies $B[A[\top], A[\top]] \Rightarrow B[A[\top], \top]$ by Monotonicity, and $B[A[\top], \top] \wedge (\top \rightarrow A[\top]) \Rightarrow B[\top, \top]$ by Formal Substitution. So BPC satisfies $A[A[\top]] \wedge (\top \rightarrow A[\top]) \Rightarrow A[\top]$. $\dashv$

IPC has the distinctive property that all its extensions are faithful. The distinctive property of FPC is Visser's Fixed Point Theorem [10]. A formula F is an *Explicit Fixed Point* of $A[P]$ over a theory Γ if $A[F] \Leftrightarrow F$ is derivable from Γ.

Theorem 2.8 (Explicit Fixed Point Theorem) *FPC satisfies* $A[A[\top]] \Leftrightarrow A[\top]$.

Proof. Apply Löb's Rule to the lemma above. $\dashv$

The converse of the Explicit Fixed Point Theorem holds too: The existence of explicit fixed points implies FPC: It even suffices that just formulas of the form $A[P] = P \to B$ have explicit fixed points. For let C be an explicit fixed point of $A[P] = P \to B$. Then $\top \to B \Rightarrow C \to B \Rightarrow C$ and, by Substitution, $C \Rightarrow \top \to B$. Thus $\top \to B$ is also an explicit fixed point of $A[P]$. So Löb's Axiom holds. Explicit fixed points need not be unique (take, for example, $A[P]$ equal to P), but sometimes they are [10]:

Proposition 2.9 *Let P be formal in $A[P]$, and let B be an explicit fixed point of $A[P]$ over FPC. Then FPC satisfies $B \Leftrightarrow A[\top]$.*

Proof. By Substitution we have $B \Rightarrow A[\top]$. Conversely, by Formal Substitution, we have $A[\top] \wedge (\top \to B) \Rightarrow A[B]$, so $A[\top] \wedge (\top \to B) \Rightarrow B$. So, by Löb's Rule, $A[\top] \Rightarrow B$. $\dashv$

The existence of single parameter explicit fixed points immediately implies the existence of multiple parameter explicit fixed points. Example: Suppose we are given formulas $A[p, q]$ and $B[p, q]$, construct formulas F and G such that FPC satisfies $F \Leftrightarrow A[F, G]$ and $G \Leftrightarrow B[F, G]$. First set $E[q]$ equal to $A[\top, q]$. So FPC satisfies $A[E[q], q] \Leftrightarrow E[q]$. If we can find G for the second equation, then F can be chosen as $E[G]$. But the second equation now looks like $B[E[G], G] \Leftrightarrow G$, so we can set G equal to $B[E[\top], \top] = B[A[\top, \top], \top]$, and thus choose F equal to $E[G] = A[\top, B[A[\top, \top], \top]]$.

The schema $\top \to A \Rightarrow A$ of IPC is a proper extension of BPC. Nonetheless there exist nontrivial formulas for which this schema holds over BPC. We even have a characterization.

Lemma 2.10 *BPC is closed under the rule*

$$\frac{((\top \to A) \to A) \to (\top \to A)}{(\top \to A) \to A \Rightarrow (\top \to A)}.$$

Proof. By the Substitution Proposition 2.4 we have $((\top \to A) \to A) \wedge (((\top \to A) \to A) \to (\top \to A)) \Rightarrow \top \to (\top \to A)$, and by Transitivity $(\top \to (\top \to A)) \wedge ((\top \to A) \to A) \Rightarrow \top \to A$. $\dashv$

To simplify notations we define $\top^n A$ recursively by $\top^0 A = A$ and $\top^{i+1} A = \top \to \top^i A$.

Given a formula A, set $C = \top A \to A$ and $D = \top A$. Then, by Lemma 2.10, BPC satisfies $C \wedge (C \to D) \Rightarrow D$.

Proposition 2.11 *Let $\Xi_A = (\top A \to A) \to \top A$. Then BPC satisfies $\top \to \Xi_A \Rightarrow \Xi_A$. Conversely, BPC is closed under the rule*

$$\frac{\top \to A \Rightarrow A}{A \Leftrightarrow (\top A \to A) \to \top A}.$$

Proof. The first statement immediately follows from Lemma 2.10 and Formalization. As to the rule, BPC obviously satisfies $A \Rightarrow (\top A \to A) \to \top A$. Conversely, suppose $\top A \Rightarrow A$ and $(\top A \to A) \to \top A$. Then, by Substitution, we have $(A \to A) \to A$, so $\top A$, and thus A. $\dashv$

Application: Let $\mathcal{S}$ be a set of formulas, and let Γ_S be axiomatized by all sequents $(\top A \to A) \to \top A$ with $A \in \mathcal{S}$. Then, by Lemma 2.10 and the faithfulness of BPC, Γ_S is faithful. Note that FPC equals $\Gamma_{\mathcal{L}}$, where $\mathcal{L}$ is the collection of all formulas.

Not all formulas of the form Ξ_A of Proposition 2.11 are derivable over BPC, but the slightly weaker nontrivial schema $((\top A \to A) \to A) \to \top A$ is, or equivalently:

Proposition 2.12 *BPC is closed under the rule*

$$\frac{A \wedge (\top B \to B) \Rightarrow B}{A \Rightarrow B}$$

Proof. By Monotonicity $A \wedge \top B \Rightarrow A \wedge (\top B \to B)$. So if we assume $A \wedge (\top B \to B) \Rightarrow B$ then, with Transitivity, $A \wedge \top B \Rightarrow B$. And thus $A \Rightarrow A \wedge (\top B \to B)$. Apply Transitivity. $\dashv$

3 Kripke Models

There are several classes of models for BPC. We give preferential attention to Kripke models, since they are easy vehicles for deriving several simple but important properties about BPC and some of its extensions. The Kripke model completeness theorem for BPC is due to Albert Visser [10]. The proof freely uses classical mathematics. We re-formulate his result and proof, since we use a sequent calculus axiomatization rather than natural deduction as in [10]. Our (essentially Kripke's, see [4]) class of Kripke models is, of course, significantly larger than the better known class of Kripke models for IPC.

A *Kripke model* is a tuple $\mathbf{K} = \langle \mathbf{W}^{\mathbf{K}}, I^{\mathbf{K}} \rangle$, where the *frame* $\mathbf{W}^{\mathbf{K}} = \mathbf{W} = (W, \prec)$ consists of a nonempty set W of *nodes*, or *worlds*, with a transitive binary relation $\prec$. The function $I^{\mathbf{K}} = I$ assigns to each atom p of the language of BPC a subset $I(p) \subseteq W$ that is forward closed, that is, if $\beta \succ \alpha \in I(p)$, then $\beta \in I(p)$.

We also write $\alpha \Vdash p$ for $\alpha \in I(p)$. The relation $\Vdash$ is uniquely extended to all formulas of the language by the inductive definition:

$$\begin{array}{lll}
\alpha \Vdash \top, & & \\
\alpha \Vdash A \wedge B & \text{if and only if} & \alpha \Vdash A \text{ and } \alpha \Vdash B, \\
\alpha \Vdash A \vee B & \text{if and only if} & \alpha \Vdash A \text{ or } \alpha \Vdash B, \\
\alpha \Vdash A \to B & \text{if and only if} & \text{for all } \beta \succ \alpha,\ \beta \Vdash A \text{ implies } \beta \Vdash B.
\end{array}$$

Let $\preceq$ be the reflexive closure of the transitive relation $\prec$. We extend the relation $\Vdash$ to all sequents by

$$\alpha \Vdash A \Rightarrow B \text{ if and only if for all } \beta \succeq \alpha,\ \beta \Vdash A \text{ implies } \beta \Vdash B.$$

A trivial induction on the complexity of formulas yields that $\beta \succeq \alpha \Vdash A$ implies $\beta \Vdash A$. So $\alpha \Vdash A$ if and only if $\alpha \Vdash (\Rightarrow A)$. A model $\mathbf{K}$ *satisfies* $A \Rightarrow B$, written $\mathbf{K} \models (A \Rightarrow B)$, if and only if $\alpha \Vdash (A \Rightarrow B)$ for all nodes $\alpha \in W$. We often write $\mathbf{K} \models A$ as short for $\mathbf{K} \models (\Rightarrow A)$. For sets of sequents Γ we write $\mathbf{K} \models \Gamma$ if and only if $\mathbf{K} \models \gamma$, for all $\gamma \in \Gamma$. We write $\Gamma \models \gamma$ when for all models $\mathbf{K}$, if $\mathbf{K} \models \Gamma$, then $\mathbf{K} \models \gamma$.

Proposition 3.1 (Soundness) *Let $\Gamma \cup \{\gamma\}$ be a set of sequents. Then $\Gamma \vdash \gamma$ implies $\Gamma \models \gamma$.*

Proof. It suffices to show that $\Vdash$ satisfies the axioms of BPC, and is closed under its rules. For example, suppose $\alpha \Vdash (A \wedge B \Rightarrow C)$, and let $\beta \succeq \alpha$ be such that $\beta \Vdash A$. Then $\gamma \Vdash B$ implies $\gamma \Vdash C$, for all $\gamma \succeq \beta$. So certainly $\beta \Vdash B \to C$. Thus $\alpha \Vdash (A \Rightarrow B \to C)$. We leave the axioms and remaining rules as easy exercises. $\dashv$

Examples: If $\mathbf{K}$ is a Kripke model with empty relation $\prec$, then $\mathbf{K} \models E_1$, where E_1 is the theory axiomatized by $\top \to \bot$. If $\mathbf{K}$ is a Kripke model with maximal relation $\prec$, then $\mathbf{K} \models$ CPC.

The Completeness Theorem, the converse of Soundness above, requires us to show that there are sufficiently many Kripke models. Below we initially construct a single Kripke model $\mathbf{U}$, essentially only dependent of the cardinality of the language. From it we construct sufficiently many models by just taking restrictions on the underlying set of worlds. A set of sequents Γ is *prime* if $\Gamma \vdash A \vee B$ implies that $\Gamma \vdash A$ or $\Gamma \vdash B$, for all A and B. Obviously, Γ is prime if and only if $\mathrm{Cl}(\Gamma)$ is prime.

Lemma 3.2 *Let $\Gamma \cup \{\gamma\}$ be a set of sequents such that $\Gamma \nvdash \gamma$. Then there exists a sequent theory $\Delta \supseteq \Gamma$ such that Δ is prime and $\Delta \nvdash \gamma$.*

Proof. Let Λ be the collection of sets of sequents $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash \gamma$. Then Λ is a nonempty set partially ordered by inclusion. Obviously, Λ is closed under unions of chains so, by Zorn's Lemma, contains a maximal element, say Δ. Clearly, Δ is a sequent theory. Suppose $\Delta \vdash A \vee B$ such that $\Delta \cup \{A\} \vdash \gamma$ and $\Delta \cup \{B\} \vdash \gamma$. We can write γ as $C \Rightarrow D$. By the Functional Completeness Proposition 2.2 we see that Δ proves the sequents $A \wedge C \Rightarrow D$ and $B \wedge C \Rightarrow D$. So $\Delta \vdash (A \vee B) \wedge C \Rightarrow D$, and thus $\Delta \vdash \gamma$; contradiction. So $\Delta \cup \{A\} \nvdash \gamma$ or $\Delta \cup \{B\} \nvdash \gamma$ hence, by maximality, $A \in \Delta$ or $B \in \Delta$. Thus Δ is prime. $\dashv$

Let $\mathbf{U}$ be the Kripke model with as set of worlds the collection $W = W^{\mathbf{U}}$ of consistent prime sequent theories. We set $\Gamma \prec \Delta$ exactly when $\Gamma \vdash A \to B$ implies $\Delta \vdash A \Rightarrow B$, for all A and B. To each atom p we assign as set $I(p) = I^{\mathbf{U}}(p)$ the collection $\{\Gamma \in W \mid p \in \Gamma\}$. Given a set of sequents Γ, let $\Gamma^{(1)} = \{A \Rightarrow B \mid \Gamma \vdash A \to B\}$. Lemma 3.2 immediately implies that $\Gamma^{(1)} = \mathrm{Cl}(\Gamma^{(1)}) = \bigcap\{\Delta \in W \mid \Gamma \prec \Delta\}$, for all $\Gamma \in W$. So $\Gamma \prec \Delta$ exactly when $\Gamma^{(1)} \subseteq \Delta$, for all $\Gamma, \Delta \in W$. Note that $\Gamma \prec \Delta$ implies $\Gamma \subseteq \Delta$. The Kripke model $\mathbf{U}$ is called the *universal model.* Its construction only depends on the cardinality of the set of atoms of the language.

Lemma 3.3 *Let Γ be a set of sequents. Then $\Gamma^{(1)} \vdash A \Rightarrow B$ if and only if $\Gamma \vdash A \to B$.*

Proof. From right to left immediately follows from the definition. The converse follows from the Formalization Proposition 2.3. $\dashv$

Lemma 3.4 *For all $\Gamma \in W^{\mathbf{U}}$ and formulas A we have $\Gamma \vdash A$, if and only if $\Gamma \Vdash A$. For all sequents γ we have $\Gamma \vdash \gamma$ implies $\Gamma \Vdash \gamma$.*

Proof. We complete the proof of both statements simultaneously, by induction on the complexity of γ. If γ equals $\Rightarrow p$ for some atom p, apply the definition of $I(p)$; the cases where $p = \top$ or $p = \bot$ are trivial. Suppose γ equals $A \Rightarrow B$ with $A \neq \top$ and $\Gamma \vdash \gamma$. By the Functional Completeness Proposition 2.2 we have $\Gamma \vdash \gamma$, if and only if $\Gamma \cup \{A\} \vdash B$. So if $\Delta \succeq \Gamma$ is such that $\Delta \vdash A$, then $\Delta \supseteq \Gamma \cup \{A\}$, and thus $\Delta \vdash B$. Apply induction: $\Gamma \Vdash \gamma$. Suppose $\Gamma \vdash A \to B$. If $\Delta \succ \Gamma$, then $\Delta \vdash A \Rightarrow B$ so, by induction, $\Delta \Vdash A \Rightarrow B$. So $\Gamma \Vdash A \to B$. Conversely, suppose $\Gamma \Vdash A \to B$. Then, by induction, $\Delta \vdash A$ implies $\Delta \vdash B$, for all $\Delta \succ \Gamma$. So $\Gamma^{(1)} \cup \{A\} \vdash B$ hence, by Functional Completeness, $\Gamma^{(1)} \vdash A \Rightarrow B$. And thus

$\Gamma \vdash A \to B$. Suppose $\Gamma \vdash A \vee B$. Then, by primality, $\Gamma \vdash A$ or $\Gamma \vdash B$. By induction, $\Gamma \Vdash A \vee B$. The converse is just as easy. We leave the remaining cases as exercises. $\dashv$

Lemma 3.4 cannot be extended to an equivalence for all sequents. Here is why the obvious proof fails: Suppose $\Gamma \Vdash A \Rightarrow B$, and suppose that $\Gamma \cup \{A\} \nvdash B$. Then there exists a consistent prime sequent theory $\Delta \supseteq \Gamma \cup \{A\}$ such that $\Delta \nvdash B$. By the Lemma, $\Delta \Vdash A$ and $\Delta \nVdash B$, hence $\Delta \nVdash A \Rightarrow B$. But we cannot guarantee that $\Delta \succeq \Gamma$, so we cannot conclude that $\Gamma \cup \{A\} \nvdash B$ leads to a contradiction. Here is an explicit example: Let p be an atom, and let Γ be a prime theory extending E_1 such that $\Gamma \nvdash p$ and $\Gamma \nvdash p \Rightarrow \bot$ (in fact, the prime theory E_1 itself will do). Then $\Gamma^{(1)}$ is inconsistent, so $\Gamma \Vdash p \Rightarrow \bot$ in the universal model $\mathbf{U}$.

For each sequent theory Γ, Let $\mathbf{U}_\Gamma$ be the model, obtained from $\mathbf{U}$, by restricting the underlying set of worlds to all $\Delta \succeq \Gamma$. A Kripke model is called *rooted* if there is a node $\alpha \in W$ such that $\alpha \prec \beta$, for all $\beta \neq \alpha$. In that case $\mathbf{K} \models \gamma$ if and only if $\alpha \Vdash \gamma$, for all sequents γ. The node α is called the *root*. If Γ is a consistent prime theory, then $\mathbf{U}_\Gamma$ is a rooted model with root Γ. The forcing relation $\Vdash$ doesn't change with respect to the nodes of $\mathbf{U}_\Gamma$, since, for all Kripke models, the only other nodes that the interpretation at a node α depends on are the nodes $\beta \succ \alpha$.

Theorem 3.5 (Completeness) *Let $\Gamma \cup \{\gamma\}$ be a set of sequents. Then $\Gamma \models \gamma$ implies $\Gamma \vdash \gamma$.*

Proof. Let γ equal $A \Rightarrow B$, and suppose $\Gamma \nvdash \gamma$. By Lemma 3.2 and Functional Completeness, there is a consistent prime sequent theory $\Delta \supseteq \Gamma \cup \{A\}$ such that $\Delta \nvdash B$. Then $\mathbf{U}_\Delta \models \Gamma$, but $\mathbf{U}_\Delta \not\models \gamma$. So $\Gamma \not\models \gamma$. $\dashv$

As an immediate consequence we have:

Theorem 3.6 (Compactness) *A set of sequents Γ has a Kripke model, if and only if each finite subset of Γ has a Kripke model.*

A sequent theory Γ is *complete* with respect to a class of Kripke models $\mathcal{K}$, if for all sequents $A \Rightarrow B$ we have $\Gamma \vdash A \Rightarrow B$, if and only if $\mathbf{K} \models A \Rightarrow B$ for all $\mathbf{K} \in \mathcal{K}$. By the Completeness Theorem 3.5 we have, for each set of sequents Γ, a class of Kripke models with respect to which Γ is complete. The theory Γ is *strongly complete* with respect to a class $\mathcal{K}$ of models, if Γ is complete with respect to $\mathcal{K}$, and if, moreover, for all sequent theories $\Delta \supseteq \Gamma$ there is a subclass of models of $\mathcal{K}$ such that Δ is complete with respect to the subclass. We sometimes write *weakly complete* instead of complete to accentuate the distinction between strong completeness and completeness.

The methods used above immediately imply:

Theorem 3.7 (Strong Completeness) *BPC is strongly complete with respect to the class of rooted Kripke models.*

Each node α of a Kripke model $\mathbf{K}$ provides us with two one-node models, $\mathbf{K}^r_\alpha$ and $\mathbf{K}^i_\alpha$, both with $\alpha \in I_\alpha(p)$ if and only if $\alpha \in I(p)$. We can make the relation $\prec_\alpha$ either reflexive or irreflexive, after which the model is completely determined. In the first case, $\mathbf{K}^r_\alpha$ is a model of Classical Propositional Calculus CPC; in the second case $\mathbf{K}^i_\alpha$ is a model of E_1, the theory axiomatized by $\top \to \bot$. Let A be a formula without implication $\to$, $\mathbf{K}$ be a Kripke model, and $\alpha \in W$. Then the inductive definition of the extension of $\Vdash$ to all formulas immediately implies that $\alpha \Vdash A$ if and only if $\mathbf{K}^r_\alpha \models A$ if and only if $\mathbf{K}^i_\alpha \models A$. Recall that a sequent $A \Rightarrow B$ is called geometric when A nor B contains implication.

Proposition 3.8 *Let $A \Rightarrow B$ be a geometric sequent, and $\mathbf{K}$ a Kripke model. To each node α assign randomly $\mathbf{K}_\alpha = \mathbf{K}^r_\alpha$ or $\mathbf{K}_\alpha = \mathbf{K}^i_\alpha$. Then $\mathbf{K} \models A \Rightarrow B$, if and only if $\mathbf{K}_\alpha \models A \Rightarrow B$, for all $\alpha \in W$.*

Proof. Let $A \Rightarrow B$ be a geometric sequent. If $\alpha \Vdash A \Rightarrow B$, then $\beta \Vdash A$ implies $\beta \Vdash B$, for all $\beta \succeq \alpha$. So $\mathbf{K}_\beta \models A$ implies $\mathbf{K}_\beta \models B$, for all $\beta \succeq \alpha$. For the converse, note that all steps above are reversible. ⊣

So the validity of geometric sequents in a model $\mathbf{K}$ is determined by the set of nodes K and its subsets $I(p)$, but is otherwise independent of the particular transitive relation $\prec$ on K.

Given a Kripke model $\mathbf{K}$ with root α, let $\mathbf{V}(\mathbf{K})$ be the model formed from $\mathbf{K}$ by adding a new root node $\alpha_0 \prec \alpha$ which is reflexive exactly when α is, and such that $\alpha_0 \Vdash p$ exactly when $\alpha \Vdash p$. The following construction is more general: Let S be a subset of $\{p \mid \alpha \Vdash p\}$. Then $\mathbf{V}_S(\mathbf{K})$ is the extension of $\mathbf{K}$ with a new root α_0 as before, except that $\alpha_0 \Vdash p$ if and only if $p \in S$.

Lemma 3.9 *Let $\mathbf{K}$ be a rooted Kripke model. If $\mathbf{K}$ has an irreflexive root, then for all S and all formulas A and B, $\mathbf{V}_S(\mathbf{K}) \models A \to B$ if and only if $\mathbf{K} \models A \Rightarrow B$. If $\mathbf{K}$ has a reflexive root, then $\mathbf{V}(\mathbf{K}) \models A \Rightarrow B$ if and only if $\mathbf{K} \models A \Rightarrow B$.*

Proof. Let α and α_0 be the irreflexive roots of $\mathbf{K}$ and $\mathbf{V}_S(\mathbf{K})$ respectively. Then obviously $\alpha_0 \vdash A \to B$ if and only if $\mathbf{K} \models A \Rightarrow B$. Suppose α is reflexive, and let α_0 be the reflexive root of $\mathbf{V}(\mathbf{K})$. It suffices to prove that $\alpha_0 \Vdash A$ if and only if $\alpha \Vdash A$, for all A. We complete the proof by induction on the complexity of A. The only nontrivial step is for A equal to $B \to C$. Obviously, if $\alpha_0 \Vdash A$, then $\alpha \Vdash A$. Suppose $\alpha \Vdash A$. From the reflexivity of α we get $\alpha \Vdash B \Rightarrow C$. By induction, $\alpha_0 \Vdash B$ if and only if $\alpha \Vdash B$, and $\alpha_0 \Vdash C$ if and only if $\alpha \Vdash C$. So $\alpha_0 \Vdash B \Rightarrow C$, and thus $\alpha_0 \Vdash B \to C$. ⊣

A theory Γ is *finitely strongly complete* with respect to a class $\mathcal{K}$ of models, if Γ is complete with respect to $\mathcal{K}$, and if, moreover, for all sequent theories $\Delta \supseteq \Gamma$ that are generated by adding finitely many sequents, there is a subclass of models of $\mathcal{K}$ such that Δ is complete with respect to the subclass.

Theorem 3.10 *Let Γ be a sequent theory, and let $\mathcal{K}$ be a class of rooted models with respect to which Γ is finitely strongly complete. Then the following are equivalent:*

(i) *Γ is faithful.*

(ii) *If $\mathbf{K} \in \mathcal{K}$ has an irreflexive root, and $\gamma \in \Gamma$ is of the form $(A_1 \to B_1) \wedge \ldots \wedge (A_n \to B_n) \Rightarrow A_0 \to B_0$, then $\mathbf{V}_S(\mathbf{K}) \models \gamma$, for some S.*

(iii) *If $\mathbf{K} \in \mathcal{K}$ has an irreflexive root, and $\delta \in \Gamma$, then $\mathbf{V}_\emptyset(\mathbf{K}) \models \delta$ or $\mathbf{V}(\mathbf{K}) \models \delta$.*

Proof. Obviously, (iii) implies (ii).

Assume (ii). To derive (i), let $\Gamma \vdash \gamma$, where $\gamma = (A_1 \to B_1) \wedge \ldots \wedge (A_n \to B_n) \Rightarrow A_0 \to B_0$, and set $\Delta = \Gamma \cup \{A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n\}$. We may assume Δ to be consistent. Let $\mathbf{K} \in \mathcal{K}$ be a model of Δ with root α. It suffices to show that $\mathbf{K} \models A_0 \Rightarrow B_0$. If α is reflexive, then this immediately follows from Γ. So assume that α is irreflexive. Consider the model $\mathbf{V}_S(\mathbf{K})$ of the condition, with new root α_0. Then $\alpha_0 \Vdash A \to B$ exactly when $\alpha \Vdash A \Rightarrow B$, for all A and B. Now $\mathbf{V}_S(\mathbf{K}) \models \gamma$, so $\alpha_0 \Vdash (A_1 \to B_1) \wedge \ldots \wedge (A_n \to B_n) \Rightarrow A_0 \to B_0$. So $\alpha \Vdash A_0 \Rightarrow B_0$, and thus $\mathbf{K} \models A_0 \Rightarrow B_0$. By finite strong completeness, $\Delta \vdash A_0 \Rightarrow B_0$.

Assume (i). To derive (iii), let $\mathbf{K} \in \mathcal{K}$ have an irreflexive root, and let $\delta \in \Gamma$. Write $\delta = A \Rightarrow B$. If $\mathbf{K} \not\models A$, then $\mathbf{V}_S(\mathbf{K}) \models A \Rightarrow B$ for all S; so we may assume that $\mathbf{K} \models A$. Additionally, we may assume that $\mathbf{V}_\emptyset(\mathbf{K}) \not\models A \Rightarrow B$. Let α_0 be the root of $\mathbf{V}_\emptyset(\mathbf{K})$. Then $\alpha_0 \Vdash A$ and $\alpha_0 \nVdash B$. Up to provable equivalence A equals a disjunction $C_1 \vee \ldots \vee C_m$ of C_i's that are conjunctions of atoms and implications. So $\alpha_0 \Vdash C$ for some $C \in \{C_1, \ldots, C_m\}$. By Transitivity, $C \Rightarrow B \in \Gamma$. Since $\alpha_0 \nVdash p$ for all atoms p, C must be equal to a conjunction of implications $(A_1 \to B_1) \wedge \ldots \wedge (A_n \to B_n)$, so $\mathbf{K} \models A_i \Rightarrow B_i$ for all i. So $\mathbf{V}_S(\mathbf{K}) \models C$ for all S. Up to provable equivalence B equals a conjunction $D_1 \wedge \ldots \wedge D_m$ of D_i's that are disjunctions of atoms and implications. Let $D \in \{D_1, \ldots, D_m\}$. It suffices to show that $\mathbf{V}(\mathbf{K}) \models D$. We have $\mathbf{K} \models C \Rightarrow D$ and, by Transitivity, $C \Rightarrow D \in \Gamma$. If $\mathbf{K} \models p$ for some atom p in the disjunction D, then $\mathbf{V}(\mathbf{K}) \models D$. Otherwise, $\mathbf{K} \not\models p$ for all atoms p in the disjunction. Replace all such leftover atoms p by implications $\top \to p$. The resulting disjunction is called E. Clearly, E is a disjunction of implications $(F_1 \to G_1) \vee \ldots \vee (F_k \to G_k)$, $C \Rightarrow E \in \Gamma$, $\mathbf{K} \models C \Rightarrow E$, and $\mathbf{V}(\mathbf{K}) \models C$. It suffices to show that $\mathbf{V}(\mathbf{K}) \models E$. If for some i we have $\mathbf{K} \not\models F_i$ and $\mathbf{K} \models F_i \to G_i$, then $\mathbf{V}(\mathbf{K}) \models E$. So we may assume that there is $m \leq k$ such that $\mathbf{K} \models F_i$ exactly when $i \leq m$, and $\mathbf{K} \not\models F_i \to G_i$ at least for all $i > m$. Now Γ proves

$$(\top \to \bigwedge_{i \leq m} F_i) \wedge C \Rightarrow \bigvee_{i \leq m} (\top \to G_i) \vee \bigvee_{i > m} (\top \to (F_i \to G_i)) \Rightarrow \top \to (\bigvee_{i \leq m} G_i \vee \bigvee_{i > m} (F_i \to G_i)).$$

By faithfulness,

$$\Gamma \cup \{\bigwedge_{i \leq m} F_i, A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n\} \vdash \bigvee_{i \leq m} G_i \vee \bigvee_{i > m} (F_i \to G_i).$$

So $\mathbf{K} \models G_i$ for some $i \leq m$, and thus $\mathbf{V}(\mathbf{K}) \models E$. $\dashv$

As a corollary we get:

Proposition 3.11 *Let Γ be a set of sequents such that its class of Kripke models is closed under the following transformation:*

> *If $\mathbf{K}$ is a rooted Kripke model of Γ with irreflexive root, then so is $\mathbf{V}_S(\mathbf{K})$, for some S.*

Then Γ is faithful.

Before we go to the main applications below, here are some illustrative applications of Proposition 3.11. An *end node* of a Kripke model is a node α such that there is no β such that $\alpha \prec \beta$. Note that reflexive nodes cannot be end nodes. First example: Let Γ_1 be axiomatized by the sequent $\top \to \bot \Rightarrow \bot$. The models of Γ_1 are exactly those without end nodes. Obviously this class satisfies the condition of Proposition 3.11. So Γ_1 is faithful. Second example: Let Γ_2 be axiomatized by the sequent $(\top \to \bot) \to \bot \Rightarrow \top \to \bot$. The models of Γ_2 are exactly those in which each node is an endnode or is below an endnode, that is, the endnodes are dense. Obviously this class satisfies the condition of Proposition 3.11. So Γ_2 is faithful. Third example: Let Γ_3 be axiomatized by all axioms of the form $p \Rightarrow \top \to \bot$, where p ranges over all atoms. Then the models of Γ_3 are exactly those for which $\alpha \nVdash p$ for all atoms p and all non-end nodes α. Obviously this class satisfies the condition of Proposition 3.11 with $S = \emptyset$. So Γ_3 is faithful.

A sequent theory Γ is *geometric* if it is axiomatizable by a set of geometric sequents.

Theorem 3.12 *Geometric theories are faithful.*

Proof. Let Γ be a geometric theory, and let $\mathbf{K}$ be a rooted model of Γ with root α. We may assume that Γ is a set of geometric sequents. For the new root α_0 of $\mathbf{V}(\mathbf{K})$, we have

$$\mathbf{K}_{\alpha_0} = \mathbf{K}_\alpha.$$

So, by Proposition 3.8, $\mathbf{V}(\mathbf{K}) \models \Gamma$. Apply Proposition 3.11. $\dashv$

Since BPC is axiomatizable by the empty set, we immediately get:

Theorem 3.13 *BPC is faithful.*

Given a sequent theory Γ, let $\Gamma_p = \mathrm{Cl}(\{A \to B \mid A \Rightarrow B \in \Gamma\})$. Then, by Faithfulness, $\Gamma = (\Gamma_p)^{(1)}$. If $\Gamma = \Psi^{(1)}$ for some sequent theory Ψ, then $\Gamma_p \subseteq \Psi$. So Γ_p is the smallest sequent theory Ψ such that $\Gamma = \Psi^{(1)}$. In fact, $\Gamma \mapsto \Gamma^{(1)}$ is the right adjoint of $\Gamma \mapsto \Gamma_p$ on the lattice of sequent theories.

Now we can show how to reduce derivability from sequent theories to derivability of a single sequent. Let $\Gamma \cup \{A \Rightarrow B\}$ be a set of sequents. Then, by Theorem 3.13, $\Gamma \vdash A \Rightarrow B$ exactly when there is a finite sequence $C_1 \Rightarrow D_1, \ldots, C_n \Rightarrow D_n \in \Gamma$ such that $(C_1 \to D_1) \wedge \ldots \wedge (C_n \to D_n) \Rightarrow A \to B$ is derivable in BPC.

Another application of Proposition 3.11 is:

Theorem 3.14 *FPC is faithful.*

Proof. Let $\mathbf{K}$ be a rooted Kripke model of FPC, and let α_0 be the root of $\mathbf{V}(\mathbf{K})$. Suppose $\alpha_0 \Vdash (\top \to A) \to A$. Then $\mathbf{K} \models (\top \to A) \Rightarrow A$ and thus, by Löb's Rule, $\mathbf{K} \models A$. So $\alpha_0 \Vdash \top \to A$. So $\mathbf{V}(\mathbf{K})$ satisfies Löb's Axiom, hence is a model of FPC. Apply Proposition 3.11. $\dashv$

A set of sequents Γ' is a *geometric extension* of a set Γ if there exists a set Δ of geometric sequents such that Γ' is axiomatizable by $\Gamma \cup \Delta$. A refined consequence of Proposition 3.11 is:

Theorem 3.15 *Let Γ be a sequent theory, and let $\mathcal{K}$ be a class of rooted models with respect to which Γ is strongly complete. Suppose that for all $\mathbf{K} \in \mathcal{K}$, if $\mathbf{K}$ has irreflexive root, then $\mathbf{V}(\mathbf{K}) \models \Gamma$. Then all geometric extensions of Γ are faithful.*

Proof. One easily verifies that if the class $\mathcal{K}$ of Kripke models of Γ satisfies the given property, then so does the subclass of Kripke models of each geometric extension. $\dashv$

In particular, all geometric extensions of FPC are faithful. Theorem 3.15 cannot be weakened by including the models $\mathbf{V}_\emptyset(\mathbf{K})$ as in Theorem 3.10. For example, let p be an atom, and let Γ and Δ be axiomatized by p and $p \Rightarrow \top \to \bot$ respectively. By Proposition 3.11, both theories are faithful, and Γ is geometric. But their union generates E_1.

Obviously the collection of sequent theories is closed under infinite intersections.

Lemma 3.16 *Let $\{\Gamma_\alpha\}_\alpha$ be a collection of sequent theories such that Γ_α is complete with respect to the class of Kripke models $\mathcal{K}_\alpha$. Then $\bigcap_\alpha \Gamma_\alpha$ is complete with respect to the class $\bigcup_\alpha \mathcal{K}_\alpha$.*

Proof. If $\bigcap_\alpha \Gamma_\alpha \vdash A \Rightarrow B$, then certainly $\Gamma_\alpha \vdash A \Rightarrow B$, for all α; so $\mathbf{K} \models A \Rightarrow B$, for all α and $\mathbf{K} \in \mathcal{K}_\alpha$. Conversely, if $\bigcap_\alpha \Gamma_\alpha \nvdash A \Rightarrow B$, then, since $\bigcap_\alpha \Gamma_\alpha$ is also closed under derivability, $\Gamma_\alpha \nvdash A \Rightarrow B$ for some α. So $\mathbf{K} \not\models A \Rightarrow B$, for some $\mathbf{K} \in \mathcal{K}_\alpha \subseteq \bigcup_\alpha \mathcal{K}_\alpha$. $\dashv$

Lemma 3.16 fails if we replace completeness by strong completeness: There is a countable collection of sequent theories $\{\Gamma_n\}_n$ satisfying $\Gamma_n \supseteq$ IPC for all n, such that Γ_n is strongly complete for the class of rooted reflexive Kripke models with at most n nodes (see, for example, [9]). Moreover, IPC is complete, but not strongly complete, for the class of finite reflexive Kripke models. So $\cap_n \Gamma_n =$ IPC is a counterexample. However, for finite intersections we have:

Proposition 3.17 *Let $\{\Gamma_\alpha\}_\alpha$ be a finite collection of sequent theories such that the intersection is faithful, and for all α, Γ_α is strongly complete with respect to the class of Kripke models $\mathcal{K}_\alpha$. Then $\bigcap_\alpha \Gamma_\alpha$ is strongly complete with respect to the class $\bigcup_\alpha \mathcal{K}_\alpha$.*

Proof. Suppose $\Gamma_\alpha \cup \{A_\beta \Rightarrow B_\beta\}_{\beta \in S} \vdash A \Rightarrow B$, for all α. Then there are finite subsets $R(\alpha)$ of S such that

$$\Gamma_\alpha \vdash \bigwedge_{\beta \in R(\alpha)} (A_\beta \to B_\beta) \Rightarrow A \to B$$

for all α. Set $P = \cup_\alpha R(\alpha)$. Then P is finite, and

$$\bigcap_\alpha \Gamma_\alpha \vdash \bigwedge_{\beta \in P} (A_\beta \to B_\beta) \Rightarrow A \to B.$$

So, by faithfulness, $\bigcap_\alpha \Gamma_\alpha \cup \{A_\alpha \Rightarrow B_\alpha\}_{\alpha \in S} \vdash A \Rightarrow B$. $\dashv$

A small twist in the completeness condition above gives:

Proposition 3.18 *Let $\{\Gamma_\alpha\}_\alpha$ be a collection of sequent theories such that the intersection is faithful, and for all α, Γ_α is finitely strongly complete with respect to the class of Kripke models $\mathcal{K}_\alpha$. Then $\bigcap_\alpha \Gamma_\alpha$ is finitely strongly complete with respect to the class $\bigcup_\alpha \mathcal{K}_\alpha$.*

Proof. As for Proposition 3.17, except that S itself is already finite. $\dashv$

Let us write $\Delta \sqsubseteq \Gamma$ whenever Γ proves all sequents and rules of Δ. Obviously, $\sqsubseteq$ is a partial order on the collection of theories. If Δ is a sequent theory, then this is equivalent to $\mathrm{Cl}(\Delta) \subseteq \mathrm{Cl}(\Gamma)$. If Δ contains closure rules, then the situation is more subtle. We illustrate this through an example. By Theorem 3.13 BPC is faithful. This implies, among other things, that if BPC $\vdash \top \to A$, then BPC $\vdash A$. Now let Γ be the theory axiomatized by the additional rule schema

$$\frac{\top \to A}{A}.$$

Then $\mathrm{Cl}(\Gamma) = \mathrm{Cl}(\emptyset)$. But Γ is a proper extension of BPC, that is, the reverse of $\emptyset \sqsubseteq \Gamma$ is false. Proof: The theory E_1, axiomatized by the axiom sequent $\top \to \bot$, is consistent, and thus a consistent extension of BPC. But it is not a consistent extension of Γ. For if E_1 were an extension of Γ, then $E_1 \vdash \bot$, so it would be inconsistent; contradiction. This example, and Propositions 2.3 and 2.2, explain why sequent theories are much easier than theories in general.

4 Special Kripke Models

A rooted Kripke model is a *tree model* if for each node the collection of predecessors forms a finite linearly ordered set. Let $\mathbf{K}$ be a rooted Kripke model, and let I be a downward closed subset of the collection of nodes of $\mathbf{K}$, that is, if $\alpha \prec \beta \in I$, then $\alpha \in I$. We construct the tree model $\mathbf{K}(I)$ as follows: As set of worlds $W(I)$ we have all sequences $\pi = (\pi_0, \ldots, \pi_n)$ of worlds $\pi_i \in W$ of $\mathbf{K}$ satisfying:

- $\pi_i \prec \pi_{i+1}$, for all i; and
- $\pi_{n-1} \in I$.

So each sequence has at most its last entry from $W \setminus I$. We define a relation $\prec_I$ on $W(I)$ by $\pi \prec_I \sigma$ if either π is a proper initial segment of σ, or π and σ have the same length, say n, such that $\pi_i = \sigma_i$ for all $i < n$, and π_n and σ_n are elements of $W \setminus I$ such that $\pi_n \prec \sigma_n$. One easily verifies that $\prec_I$ is a transitive relation. Note that if $I = \emptyset$, then the transitive sets $(W, \prec)$ and $(W(I), \prec_I)$ are isomorphic in the obvious way. Define the forcing relation $\Vdash_I$ on $\mathbf{K}(I)$ by $\pi \Vdash_I p$ if and only if $\pi_n \Vdash p$, for all $\pi = (\pi_0, \ldots, \pi_n)$ and atoms p.

Lemma 4.1 *Let* $\mathbf{K}$ *be a Kripke model, and let* I *be a downward closed subset of the collection of nodes of* $\mathbf{K}$. *Then for each* $\pi = (\pi_0, \ldots, \pi_n) \in W(I)$, *and each sequent* γ, *we have* $\pi \Vdash_I \gamma$, *if and only if* $\pi_n \Vdash \gamma$.

Proof. The statement is clearly valid for all $\pi = (\pi_0, \ldots, \pi_n)$ where $\pi_n \in W \setminus I$. Moreover, it suffices to prove the statement for formulas A only. We complete the proof by induction on the complexity of A. The case for atoms follows from the definition, and the cases for $\top$ and $\bot$ are trivial. The induction steps where A is a conjunction or a disjunction are also easy. Let $A = B \to C$ and $\pi = (\pi_0, \ldots, \pi_n)$ with $\pi_n \in I$. If $\pi \not\Vdash_I A$, then there is a sequence π' properly extending π, with last entry λ, such that $\pi' \Vdash_I B$ and $\pi' \not\Vdash_I C$. So $\pi_n \prec \lambda$. By induction, $\lambda \Vdash B$ and $\lambda \not\Vdash C$. So $\pi_n \not\Vdash B \to C$. Conversely, suppose $\pi_n \not\Vdash B \to C$. There exists $\lambda \succ \pi_n$ such that $\lambda \Vdash B$ and $\lambda \not\Vdash C$. Set $\pi' = (\pi_0, \ldots, \pi_n, \lambda) \succ_I \pi$. By induction, $\pi' \Vdash_I B$ and $\pi' \not\Vdash_I C$. So $\pi \not\Vdash_I B \to C$. ⊣

Corollary 4.2 *BPC is strongly complete for the class of irreflexive trees.*

Proof. Apply Lemma 4.1 with $I = W$. ⊣

Underlying each Kripke model $\mathbf{K} = \langle \mathbf{W}^{\mathbf{K}}, I^{\mathbf{K}} \rangle$ is a frame $\mathbf{W}^{\mathbf{K}} = \mathbf{W} = (W, \prec)$ consisting of a set W of *nodes*, or *worlds*, with a transitive binary relation $\prec$. A transitive set is turned into a Kripke model by adding an assignment function $I = I^{\mathbf{K}}$ or, equivalently, by adding a forcing relation $\Vdash$. Let Γ be a set of sequents. We write $(W, \prec) \models \Gamma$, that is, $(W, \prec)$ *models* Γ, if $\mathbf{K} \models \Gamma$, for all Kripke models $\mathbf{K}$ with $(W, \prec)$ as underlying transitive set. Note that, despite what Corollary 4.2 may suggest, there is no transitive set $(W \prec)$ with an irreflexive node such that $(W, \prec) \models$ IPC. Let $(W_0, \prec_0)$ and $(W_1, \prec_1)$ be two transitive sets. A function $f : W_0 \to W_1$ is a *p-morphism* of transitive sets if the following conditions hold:

- f is a morphism, that is, $f(x) \prec_1 f(y)$ whenever $x \prec_0 y$; and
- for all $x \in W_0$ and $y \in W_1$, if $f(x) \prec_1 y$, then there exists $z \in W_0$ such that $x \prec_0 z$ and $f(z) = y$.

The following, when combined with an embedding into K4, essentially is a special case of [8]:

Proposition 4.3 *Let* f *be a p-morphism from* $(W_0, \prec_0)$ *onto* $(W_1, \prec_1)$, *and let* $\Vdash_1$ *be a forcing relation on* $(W_1, \prec_1)$. *Define* $\Vdash_0$ *on* $(W_0, \prec_0)$ *by* $x \Vdash_0 p$ *exactly when* $f(x) \Vdash_1 p$, *for atomic* p. *Then* $\Vdash_0$ *is a forcing relation on* $(W_0, \prec_0)$ *such that for all* $x \in W_0$ *and sequents* γ *we have* $x \Vdash_0 \gamma$ *if and only if* $f(x) \Vdash_1 \gamma$.

Proof. From f being order preserving we immediately see that $\Vdash_0$ is a forcing relation on $(W_0, \prec_0)$. We complete the proof by induction on the complexity of γ. First the cases where γ equals $\top \Rightarrow B$. The induction steps for B atomic, a conjunction, or a disjunction, are trivial. Let B equal $C \to D$. Suppose $x \Vdash_0 C \to D$, and $f(x) \prec_1 y \Vdash_1 C$. To show: $y \Vdash_1 D$. But $y = f(z)$ for some $z \succ_0 x$ so, by induction, $z \Vdash_0 C$. So $z \Vdash_0 D$ and thus, again by induction, $y = f(z) \Vdash_1 D$. Conversely, suppose $f(x) \Vdash_1 C \to D$, and $x \prec_0 z \Vdash_0 C$. Then, by induction, $f(x) \prec_1 f(z) \Vdash_1 C$, so also $f(z) \Vdash_1 D$. Thus, again by induction, $z \Vdash_0 D$. Next the cases where γ equals $A \Rightarrow B$ with $A \neq \top$. Suppose $x \Vdash_0 A \Rightarrow B$, and $f(x) \preceq_1 y \Vdash_1 A$. To show: $y \Vdash_1 B$. The further proof is identical to the one above, with A and B replacing C and D. Conversely, suppose $f(x) \Vdash_1 A \Rightarrow B$, and $x \preceq_0 z \Vdash_0 A$. Again the proof proceeds exactly along the lines above, with A and B replacing C and D. $\dashv$

As a corollary we get:

Proposition 4.4 *Let Γ be a set of sequents, and let f be a p-morphism from $(W_0, \prec_0)$ onto $(W_1, \prec_1)$. If $(W_0, \prec_0) \models \Gamma$, then $(W_1, \prec_1) \models \Gamma$.*

Proof. Suppose $\langle (W_1, \prec_1), \Vdash_1 \rangle \not\models \Gamma$, and let $\Vdash_0$ be the forcing relation on $(W_0, \prec_0)$ as in Proposition 4.3. Then $\langle (W_0, \prec_0), \Vdash_0 \rangle \not\models \Gamma$. $\dashv$

Application: Let $(W, \prec)$ be a nonempty transitive set such that for all $x \in W$ there exists some $y \succ x$. Then if $(W, \prec) \models \Gamma$, then $\Gamma \cup \mathrm{CPC}$ is consistent. Proof: Take the obvious p-morphism from $(W, \prec)$ onto the singleton reflexive transitive set, and apply Proposition 4.4.

Given a set of sequents Γ, define $\mathrm{Sub}(\Gamma)$ to be the set of sequents $A \Rightarrow B$ for which there are sequents $A_1 \Rightarrow A_2$ and $B_1 \Rightarrow B_2$ in Γ such that A equals $\top$ or is a subformula of one of the A_i, and B is a subformula of one of the B_i. The transformation $\mathbf{K} \mapsto \mathbf{K}_\Gamma$ below reduces Kripke models in such a way that essentially only the necessary structure involving the set Γ is left. Given a node α of $\mathbf{K}$, define $[\alpha]_\Gamma = [\alpha] \subseteq \mathrm{Sub}(\Gamma)$ by $[\alpha] = \{A \Rightarrow B \in \mathrm{Sub}(\Gamma) \mid \alpha \Vdash A \Rightarrow B\}$. Set $W_\Gamma = \{[\alpha] \mid \alpha \in W\}$, and $[\alpha] \prec_\Gamma [\beta]$ if and only if $[\alpha] \subseteq [\beta]$ and, additionally, $A \to B \in [\alpha]$ and $A \in [\beta]$ imply $B \in [\beta]$, for all A and B. Clearly, $\prec_\Gamma$ is a transitive relation on W_Γ. For atomic p set $[\alpha] \Vdash_\Gamma p$ if and only if $p \in [\alpha]$. This makes $\mathbf{K}_\Gamma = \langle (W_\Gamma, \prec_\Gamma), \Vdash_\Gamma \rangle$ a Kripke model over the sublanguage $\mathcal{L}_\gamma$ whose atoms are just those that occur in formulas of Γ. The map $\alpha \mapsto [\alpha]$ is a morphism of $(W, \prec)$ onto $(W_\Gamma, \prec_\Gamma)$. If $\mathbf{K}$ is a rooted model, then so is $\mathbf{K}_\Gamma$. We call $\mathbf{K}_\Gamma$ the *minimal model* of $\mathbf{K}$ relative to Γ. If Γ equals the class of all sequents of the language $\mathcal{L}$, then we call $\mathbf{K}_\Gamma = \mathbf{K}_\mathcal{L}$ just the *minimal model* of $\mathbf{K}$.

Proposition 4.5 *Let $\mathbf{K}$ be a Kripke model, and let Γ be a set of sequents. Then for all nodes α and sequents $\gamma \in Sub(\Gamma)$ the following are equivalent:*

- *$\alpha \Vdash \gamma$;*
- *$\gamma \in [\alpha]$; and*
- *$[\alpha] \Vdash_\Gamma \gamma$.*

Proof. The first and second items are equivalent by definition. We complete the proof by induction on the complexity of γ. First the cases where γ equals $\top \Rightarrow B$. The induction steps for B atomic, a conjunction, or a disjunction, are trivial. Let B equal $C \to D$. Suppose $\alpha \Vdash C \to D$, and $[\alpha] \prec_\Gamma [\beta] \Vdash_\Gamma C$. To show: $[\beta] \Vdash_\Gamma D$. By induction, $\beta \Vdash C$. But $C \to D \in [\alpha]$, so $D \in [\beta]$. And so $\beta \Vdash D$ and thus, again by induction, $[\beta] \Vdash_\Gamma D$. Conversely, suppose $[\alpha] \Vdash_\Gamma C \to D$, and $\alpha \prec \beta \Vdash C$. To show: $\beta \Vdash D$. By induction, $[\alpha] \prec_\Gamma [\beta] \Vdash_\Gamma C$, so also $[\beta] \Vdash_\Gamma D$. Thus, again by induction, $\beta \Vdash D$. Next the cases where γ equals $A \Rightarrow B$ with $A \neq \top$. Suppose $\alpha \Vdash A \Rightarrow B$, and $[\alpha] \preceq_\Gamma [\beta] \Vdash_\Gamma A$. To show: $[\beta] \Vdash_\Gamma B$. The further proof is identical to the one above, with A and B replacing C and D. Conversely, suppose $[\alpha] \Vdash_\Gamma A \Rightarrow B$, and $\alpha \preceq \beta \Vdash A$. Again the proof proceeds exactly along the lines above, with A and B replacing C and D. $\dashv$

A Kripke model is *finite* if its underlying transitive set is finite. If the set of sequents Γ is finite, then $\mathbf{K}_\Gamma$ is finite, for all Kripke models $\mathbf{K}$. So Proposition 4.5 immediately implies

Theorem 4.6 *BPC is complete with respect to the class of finite Kripke models.*

BPC is not strongly complete with respect to the class of finite Kripke models. For example, let $p_0, p_1, p_2, \ldots$ be a countably infinite sequence of atoms, and let $\Gamma = \{(p_i \leftrightarrow p_j) \Rightarrow p_0 \mid 0 < i < j\}$. If $\mathbf{K}$ is a finite model of Γ, then $\mathbf{K} \models p_0$; but $\Gamma \not\vdash p_0$. For a different proof of Theorem 4.6, see [10]. The Theorem and its proof also allow us to show that derivability in BPC is decidable, see [10].

Proposition 4.7 *For all n, let F_n be the sequent theory axiomatized by $\top^{n+1}\bot \Rightarrow \top^n\bot$. Then $F_m \subseteq F_n$ whenever $m \geq n$, and $\cap_n F_n = BPC$.*

Proof. Obviously, $F_n \supseteq F_{n+1}$. Additionally, F_n holds in all models where the longest properly ascending path has length at most n. So $\cap_n F_n$ is valid in all finite Kripke models, hence equals BPC. $\dashv$

Even if $\mathbf{K}_\Gamma$ is finite, we don't have that for each node $[\alpha]_\Gamma$ there is a formula A such that $[\beta]_\Gamma \Vdash_\Gamma A$ if and only if $[\alpha]_\Gamma \preceq_\Gamma [\beta]_\Gamma$. For example, let $\mathbf{K}$ have as underlying transitive set two nodes, α and β, such that α is reflexive and β is irreflexive, and no other relations between the nodes. Set $\alpha \Vdash p$ if and only if $\beta \Vdash p$, for all atoms p. Suppose that $\alpha \Vdash A$. Up to provable equivalence we may assume A to be a disjunction $A_1 \vee \ldots \vee A_n$ of conjunctions A_i of atoms and implications. So $\alpha \Vdash A_i$ for some i. Now $\beta \Vdash p$ for all atoms p of this conjunction, and $\beta \Vdash \top \to \bot$, so β also satisfies all implications. So $\beta \Vdash A_i$, and thus $\beta \Vdash A$. But we do have:

Lemma 4.8 *Let $\mathbf{K}$ be a Kripke model, and let Γ be a set of sequents. Let $[\alpha] \neq [\beta]$ be nodes of $\mathbf{K}_\Gamma$. Then there exists a formula A such that $[\alpha] \Vdash_\Gamma A$ and $[\beta] \not\Vdash_\Gamma A$, or $[\alpha] \not\Vdash_\Gamma A$ and $[\beta] \Vdash_\Gamma A$. Moreover, either $A \in \Gamma$, or A equals $A_1 \to A_2$ such that $A_1 \Rightarrow A_2 \in \Gamma$.*

Proof. Given $[\alpha] \neq [\beta]$, there is a sequent $\gamma \in \mathrm{Sub}(\Gamma)$ such that $[\alpha] \Vdash_\Gamma \gamma$ and $[\beta] \not\Vdash_\Gamma \gamma$, or $[\alpha] \not\Vdash_\Gamma \gamma$ and $[\beta] \Vdash_\Gamma \gamma$. By symmetry we may assume that $[\alpha] \not\Vdash_\Gamma \gamma$ and $[\beta] \Vdash_\Gamma \gamma$. There are formulas C and D such that γ equals $C \Rightarrow D$. If $[\alpha] \not\Vdash_\Gamma C \to D$, then we can choose A equal to $C \to D$; so we may assume that $[\alpha] \Vdash_\Gamma C \to D$. So $[\alpha] \Vdash_\Gamma C$ and $[\alpha] \not\Vdash_\Gamma D$. If $[\beta] \Vdash_\Gamma C$, then set A equal to D. Otherwise, set A equal to C. $\dashv$

Lemma 4.9 *Let Γ be a set of sequents, and $\mathcal{K}$ a class of Kripke models of Γ such that for every rooted Kripke model $\mathbf{K}$ of Γ we have $\mathbf{K}_L \in \mathcal{K}$. Then Γ is strongly complete with respect to $\mathcal{K}$. More generally, let $\mathcal{J}$ and $\mathcal{K}$ be classes of rooted Kripke models of Γ such that Γ is (strongly) complete with respect to $\mathcal{J}$. Suppose that $\mathbf{K}_L \in \mathcal{K}$, for all $\mathbf{K} \in \mathcal{J}$. Then Γ is (strongly) complete with respect to $\mathcal{K}$.*

Proof. We only prove the first claim. Let $\Delta \cup \{\gamma\}$ be a set of sequents such that $\Delta \supseteq \Gamma$ and $\Delta \not\vdash \gamma$. Then there is a rooted Kripke model $\mathbf{K} \models \Delta$ such that $\mathbf{K} \not\models \gamma$. Then, by Proposition 4.5, we have $\mathbf{K}_L \models \Gamma$ and $\mathbf{K}_L \not\models \gamma$. $\dashv$

Let LBPC be the extension of BPC with the schema $(A \to B) \vee ((A \to B) \to A)$. A Kripke model is *linear* if for all pairs of nodes α and β we have $\alpha \prec \beta$ or $\alpha = \beta$ or $\alpha \succ \beta$. Albert Visser proved in [10]:

Theorem 4.10 *LBPC is strongly complete with respect to the class of rooted linear Kripke models.*

Proof. One easily verifies that all linear models are models of LBPC. Let $\mathbf{K}$ be a rooted Kripke model of LBPC with root α_0, and let $[\alpha] \neq [\beta]$ be two nodes of the minimal model $\mathbf{K}_L$. By Lemma 4.8 we may assume that there is a formula A such that $[\alpha] \not\Vdash_L A$ and $[\beta] \Vdash_L A$. To show: $[\alpha] \prec_L [\beta]$. We may assume that $[\alpha] \neq [\alpha_0]$, for $[\alpha_0]$ is the root of $\mathbf{K}_L$; so $[\alpha_0] \prec_L [\alpha]$. Suppose $[\alpha] \Vdash_L B \to C$ and $[\beta] \Vdash_L B$. We must show that $[\beta] \Vdash_L C$. There are two possibilities. First suppose $[\alpha] \not\Vdash B$. Then $[\alpha_0] \prec_L [\alpha] \not\Vdash_L B \to C \Rightarrow B$. So, by LBPC, $[\beta] \succ_L [\alpha_0] \Vdash_L B \to C$, and thus $[\beta] \Vdash_L C$. Otherwise, suppose $[\alpha] \Vdash_L B$. Then $[\alpha] \Vdash_L \top \to C$, hence also $[\alpha] \Vdash_L A \to C$. Then $[\alpha_0] \prec_L [\alpha] \not\Vdash_L A \to C \Rightarrow A$. So, by LBPC, $[\beta] \succ_L [\alpha_0] \Vdash_L A \to C$, and thus $[\beta] \Vdash_L C$. Next we must show $[\alpha] \subseteq [\beta]$. Let $B \in [\alpha]$. Then $[\alpha] \Vdash_L B$, so $[\alpha_0] \not\Vdash_L B \to A$. By LBPC, therefore, $[\alpha_0] \Vdash_L (B \to A) \to B$. Since $[\alpha_0] \prec_L [\beta] \Vdash_L B \to A$, we have $[\beta] \Vdash_L B$. Finally, let $B \Rightarrow C \in [\alpha]$. Then $B \to C \in [\beta]$. Suppose $B \in [\beta]$. Then, as above, $C \in [\beta]$. So $B \Rightarrow C \in [\beta]$. And thus $[\alpha] \subseteq [\beta]$. $\dashv$

Intuitionistic Propositional Calculus IPC is axiomatized by BPC plus the sequent schema $\top \to A \Rightarrow A$. The following is well-known, and can also be shown using the minimal model construction:

Theorem 4.11 *IPC is strongly complete with respect to the class of reflexive rooted Kripke models.*

Proof. Obviously, all reflexive models are models of IPC. Let $\mathbf{K}$ be a rooted Kripke model of IPC with root α_0. So $[\alpha_0]$ is the root of $\mathbf{K}_L$. Let $[\alpha]$ be a node. Clearly, $[\alpha] \subseteq [\alpha]$. Suppose that $A \to B \in [\alpha]$ and $A \in [\alpha]$. Then $\top \to B \in [\alpha]$ thus also, since $\mathbf{K}_L$ is a model of IPC, $B \in [\alpha]$. So $[\alpha] \prec_L [\alpha]$. $\dashv$

As an easy consequence we get:

Corollary 4.12 *IPC is complete with respect to the class of finite reflexive rooted Kripke models.*

Proof. If $\mathbf{K}$ is reflexive, then so is $\mathbf{K}_\Gamma$. So if, additionally, Γ is finite, then $\mathbf{K}_\Gamma$ is a finite reflexive model. $\dashv$

In [10] the theorem below is also proved.

Theorem 4.13 *FPC is complete with respect to the class of finite rooted irreflexive Kripke models.*

Proof. Clearly, each finite irreflexive Kripke model is a model of FPC. Let FPC $\not\vdash \gamma$, for some sequent γ. There is a rooted Kripke model $\mathbf{K}$ of FPC such that $\mathbf{K} \not\models \gamma$. Let Γ be the finite collection of sequents generated by γ. Then $\mathbf{K}_\Gamma$ is a finite rooted model such that $\mathbf{K}_\Gamma \not\models \gamma$. Let $\mathbf{K}_i$ be the finite model obtained from $\mathbf{K}_\Gamma$ by removing the diagonal from $\prec_\Gamma$, that is, $[\alpha] \prec_i [\beta]$ exactly when both $[\alpha] \prec_\Gamma [\beta]$ and $[\alpha] \neq [\beta]$. Then $\mathbf{K}_i$ is a finite irreflexive rooted Kripke model of FPC. It suffices to show that $\mathbf{K}_i \models \delta$ if and only if $\mathbf{K}_\Gamma \models \delta$, for all sequences $\delta \in \Gamma$. We complete the proof by induction on the complexity of δ. Since $\preceq_i$ and $\preceq_\Gamma$ are equal, the only nontrivial induction step is for $\to$. Suppose $[\alpha] \Vdash_\Gamma A \to B$. Then $[\beta] \Vdash_\Gamma A$ implies $[\beta] \Vdash_\Gamma B$, for all $[\beta] \succ_\Gamma [\alpha]$. If $[\beta] \succ_i [\alpha]$, then certainly $[\beta] \succ_\Gamma [\alpha]$ so, by induction, $[\beta] \succ_i [\alpha]$ and $[\beta] \Vdash_i A$ imply $[\beta] \Vdash_i B$. Thus $[\alpha] \Vdash_i A \to B$. Conversely, suppose $[\alpha] \nVdash_\Gamma A \to B$. To show: $[\alpha] \nVdash_i A \to B$. The only difficult case occurs when $[\alpha] \Vdash_\Gamma A$, $[\alpha] \nVdash_\Gamma B$, $[\alpha] \prec_\Gamma [\alpha]$, and $[\beta] \Vdash_\Gamma B$ for all $[\beta] \succ_i [\alpha]$. Let $\alpha \prec \beta_1 \preceq \beta_2 \Vdash A \to B$ in the original model $\mathbf{K}$. Since $A \to B \in \Gamma$, we have $[\alpha] \prec_i [\beta_2]$, hence $[\beta_2] \Vdash_\Gamma B \in \Gamma$. So $\beta_2 \Vdash B$. Therefore $\beta_1 \Vdash A \to B \Rightarrow B$. So $\alpha \Vdash (A \to B) \to B$. But $\mathbf{K}$ is a model of FPC, so it satisfies $(A \to B) \to B \Rightarrow (\top \to B) \to B \Rightarrow \top \to B \Rightarrow A \to B$. So $\alpha \Vdash A \to B$, and thus $[\alpha] \Vdash A \to B$; contradiction. $\dashv$

As shown in [10], all models whose underlying transitive sets are irreflexive and co-wellfounded, are models of FPC.

Proposition 4.14 *For all n, let E_n be the sequent theory axiomatized by $\top^n \bot$. Then $E_m \subseteq E_n$ whenever $m \geq n$, and $\cap_n E_n =$ FPC.*

Proof. Obviously, $E_n \supseteq E_{n+1} \supseteq$ FPC. Additionally, E_n holds in all irreflexive models where the longest properly ascending path has length at most n. So $\cap_n E_n$ is valid in all finite irreflexive Kripke models, hence equals FPC. $\dashv$

Classical Propositional Calculus CPC is the extension of IPC that is axiomatizable by the schema $A \vee \neg A$. We leave it as an easy exercise to prove the following well-known result using minimal models.

Theorem 4.15 *CPC is strongly complete with respect to the class of one-node reflexive models.*

Recall that E_1 is the extension of BPC axiomatizable by $\top \to \bot$. The following doesn't need the minimal model construction, but it falls in the same category of results.

Theorem 4.16 *E_1 is strongly complete with respect to the class of one-node irreflexive models.*

Proof. Obviously, all irreflexive singleton models are models of E_1. Let $\mathbf{K}$ be a rooted Kripke model of E_1 with root $\alpha_0 \Vdash \top \to \bot$. If there were an other node β, then $\alpha_0 \prec \beta \Vdash \bot$, contradiction. So α_0 is the only node. If $\alpha_0 \prec \alpha_0$, then $\alpha_0 \Vdash \bot$, contradiction. So $\mathbf{K}$ is already a one-node irreflexive model. $\dashv$

5 Intermediate Logics

Each assignment τ from the atoms to the collection of all formulas extends to a (substitution) map $B \mapsto \tau B$ from the language to itself by replacing all occurrences of atoms p in a formula B by τp. A theory Γ is *schematic* if $\Gamma \vdash A \Rightarrow B$ implies $\Gamma \vdash \tau A \Rightarrow \tau B$, for all substitutions τ. An *intermediate logic* is a consistent schematic sequent theory. The theories BPC, IPC, CPC, FPC, and E_1 are all intermediate logics. Below we prove that CPC and E_1 are maximal among the intermediate logics ordered by $\subseteq$.

Lemma 5.1 *Let $\Gamma \cup \{\top \to \bot \Rightarrow \bot\}$ be a consistent set of sequents. Then $\Gamma \cup CPC$ is consistent.*

Proof. We may assume that $\Gamma \vdash \top \to \bot \Rightarrow \bot$. We first show that $\Gamma \cup \mathrm{IPC}$ is consistent. Let $A_1, \ldots, A_n$ be a finite sequence of formulas. By the Compactness Theorem 3.6, it suffices to show that $\Delta = \Gamma \cup \{\top \to A_1 \Rightarrow A_1, \ldots, \top \to A_n \Rightarrow A_n\}$ is consistent. Let $\mathbf{K}$ be a Kripke model of Γ. Then for all nodes α there is a node $\beta \succ \alpha$. We construct an ascending sequence of nodes $\alpha_0, \ldots, \alpha_n$ as follows. Choose α_0 arbitrarily. Suppose α_{i-1} has been chosen. If $\alpha_{i-1} \Vdash \top \to A_i \Rightarrow A_i$, set α_i equal to any node $\beta \succeq \alpha_{i-1}$. Otherwise, there is $\beta \succeq \alpha_{i-1}$ such that $\beta \Vdash \top \to A_i$. Set α_i equal to any node $\gamma \succ \beta$. Then $\alpha_n \Vdash \Delta$.

Next, we may assume that $\Gamma \vdash \mathrm{IPC}$. Let $A_1, \ldots, A_n$ be a finite sequence of formulas. By the Compactness Theorem 3.6, it suffices to show that the set $\Delta = \Gamma \cup \{A_1 \vee \neg A_1, \ldots, A_n \vee \neg A_n\}$ is consistent. Let $\mathbf{K}$ be a Kripke model of Γ. For all nodes α there are nodes $\beta \succ \alpha$. We construct an ascending sequence of nodes $\alpha_0, \ldots, \alpha_n$ as follows. Choose α_0 arbitrarily. Suppose α_{i-1} has been chosen. If $\alpha_{i-1} \Vdash \neg A_i$, set α_i equal to any node $\beta \succeq \alpha_{i-1}$. Otherwise, there is $\beta \succ \alpha_{i-1}$ such that $\beta \Vdash A_i$. Set α_i equal to any node $\gamma \succeq \beta$. Then $\alpha_n \Vdash \Delta$. So $\Gamma \cup \mathrm{CPC}$ is consistent. $\dashv$

The following, when combined with an embedding into K4, essentially is a special case of [5]:

Theorem 5.2 *CPC and E_1 are the only maximal theories among the intermediate logics over a fixed language, ordered by $\subseteq$. Each intermediate logic over this language is contained in CPC or in E_1.*

Proof. First we show that CPC and E_1 are maximal. Let Γ be an intermediate logic, and let $A \Rightarrow B$ be a sequent. Suppose $\mathrm{CPC} \subseteq \Gamma$, and $\mathrm{CPC} \nvdash A \Rightarrow B$. So there is a one-node reflexive model $\mathbf{K}$ such that $\mathbf{K} \not\models A \Rightarrow B$. Then there is a substitution map τ, assigning $\top$ or $\bot$ to all atoms, such that CPC proves both $\tau A \Leftrightarrow \top$ and $\tau B \Leftrightarrow \bot$. Then $\Gamma \nvdash \tau A \Rightarrow \tau B$, so, since Γ is schematic, $\Gamma \nvdash A \Rightarrow B$. So $\Gamma = \mathrm{CPC}$. Next, suppose that $E_1 \subseteq \Gamma$ and $E_1 \nvdash A \Rightarrow B$. There is a one-node irreflexive model $\mathbf{K}$ such that $\mathbf{K} \not\models A \Rightarrow B$. So there is a substitution map τ, assigning $\top$ or $\bot$ to all atoms, such that E_1 proves both $\tau A \Leftrightarrow \top$ and $\tau B \Leftrightarrow \bot$. Then $\Gamma \nvdash \tau A \Rightarrow \tau B$, so, since Γ is schematic, $\Gamma \nvdash A \Rightarrow B$. So $\Gamma = E_1$.

Now let Γ be an arbitrary intermediate logic. If $\Gamma \cup E_1$ is consistent, then $\Gamma \subseteq E_1$. Otherwise, $\Gamma \vdash \top \to \bot \Rightarrow \bot$ is consistent so, by Lemma 5.1, $\Gamma \cup \mathrm{CPC}$ is consistent. So $\Gamma \subseteq \mathrm{CPC}$. $\dashv$

We have the following dual to Lemma 5.1:

Proposition 5.3 *Let $\Gamma \cup \{(\top \to \bot) \to \bot \Rightarrow \top \to \bot\}$ be a consistent set of sequents. Then $\Gamma \cup E_1$ is consistent.*

Proof. Suppose $\Gamma \cup E_1$ is inconsistent. Then $\Gamma \vdash (\top \to \bot) \to \bot$. But then $\Gamma \cup \{(\top \to \bot) \to \bot \Rightarrow \top \to \bot\} \vdash \top \to \bot$, contradiction. $\dashv$

The theories $\mathrm{Cl}\{\top \to \bot \Rightarrow \bot\}$ and $\mathrm{Cl}\{(\top \to \bot) \to \bot \Rightarrow \top \to \bot\}$ are also the smallest sequent theories that are relatively inconsistent with E_1 and CPC respectively:

Proposition 5.4 *Let Γ be a sequent theory. Then $\Gamma \cup E_1$ is inconsistent exactly when $\Gamma \vdash \top \to \bot \Rightarrow \bot$. And $\Gamma \cup CPC$ is inconsistent exactly when $\Gamma \vdash (\top \to \bot) \to \bot \Rightarrow \top \to \bot$.*

Proof. The case for $\Gamma \cup E_1$ is obvious. By Lemma 5.1 $\Gamma \cup$CPC is consistent if and only if $\Gamma \cup \{\top \to \bot \Rightarrow \bot\}$ is consistent. If $\Gamma \cup \{\top \to \bot \Rightarrow \bot\}$ is inconsistent then, by the Formalization Proposition 2.3, $\Gamma \vdash (\top \to \bot) \to \bot \Rightarrow \top \to \bot$. Conversely, if $\Gamma \vdash (\top \to \bot) \to \bot \Rightarrow \top \to \bot$ then, by Implication Introduction and Transitivity, $\Gamma \cup \{\top \to \bot \Rightarrow \bot\} \vdash \top \to \bot$ and thus, again by Transitivity, $\Gamma \cup \{\top \to \bot \Rightarrow \bot\} \vdash \bot$. $\dashv$

The Kripke models of $\top \to \bot \Rightarrow \bot$ are exactly those that don't have endnodes. The Kripke models of $(\top \to \bot) \to \bot \Rightarrow \top \to \bot$ (or, by Proposition 2.11, of $\Xi_\bot$) are exactly those where each node is an end node or is below an endnode.

Obviously the collection of intermediate logics over a fixed language is closed under infinite intersections. A difference between the case for IPC and the case for BPC is, that over IPC all intermediate logics are contained in CPC; so the collection is a complete lattice. But the collection of intermediate logics over BPC is not even closed under finite joins, for CPC $\cup E_1$ is inconsistent. Theorem 5.2 implies that the lattice of intermediate logics almost is.

A simple consequence of Proposition 3.8 and Theorem 5.2 is the following conservativity result.

Proposition 5.5 *Let Γ be an intermediate logic, and let $A \Rightarrow B$ be a geometric sequence. Then $\vdash A \Rightarrow B$ if and only if $\Gamma \vdash A \Rightarrow B$.*

Proof. Let $A \Rightarrow B$ be a geometric sequent such that $\nvdash A \Rightarrow B$. By Theorem 5.2 it suffices to show that CPC $\nvdash A \Rightarrow B$, and $E_1 \nvdash A \Rightarrow B$. Let $\mathbf{K}$ be a Kripke model such that $\mathbf{K} \not\models A \Rightarrow B$. Then, by Proposition 3.8, there is a node α such that $\mathbf{K}^r_\alpha \not\models A \Rightarrow B$ and $\mathbf{K}^i_\alpha \not\models A \Rightarrow B$. But $\mathbf{K}^r_\alpha$ is a model of CPC, and $\mathbf{K}^i_\alpha$ is a model of E_1. $\dashv$

So the only geometric intermediate logic is the minimal one: BPC.

Recall that, for sets of sequents Γ, we defined $\Gamma^{(1)} = \{A \Rightarrow B \mid \Gamma \vdash A \to B\}$. We call a sequent theory Γ *purely formal* if it is axiomatizable by a set of sequents of the form $A \to B$.

Theorem 5.6 *Let Γ_1, Γ_2 be sets of sequents such that $\Gamma_2 \subseteq \Delta$, where Δ is a purely formal sequent theory satisfying $\Delta^{(1)} \subseteq Cl(\Gamma_1 \cup \Gamma_2)$. Then $Cl(\Gamma_1) \cap Cl(\Gamma_2)$ is axiomatizable by the collection of axioms*

$$A \Rightarrow B \vee (C \to D);$$

$$C \Rightarrow D \vee (A \to B); \quad \textit{and}$$

$$(A \to B) \vee (C \to D),$$

where $A \Rightarrow B$ ranges over Γ_1 and $C \Rightarrow D$ ranges over Γ_2.

Proof. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the classes of rooted Kripke models of Γ_1 and Γ_2 respectively, and let Λ be the collection of sequents derived from Γ_1 and Γ_2 as described above. Obviously, $\mathbf{K} \models \lambda$, for all $\mathbf{K} \in \mathcal{K}_1 \cup \mathcal{K}_2$ and $\lambda \in \Lambda$. Conversely, let $\mathbf{K}$ be a rooted Kripke model with root α such that $\mathbf{K} \models \lambda$, for all $\lambda \in \Lambda$. It suffices to show that $\mathbf{K} \in \mathcal{K}_1 \cup \mathcal{K}_2$. We may assume that $\mathbf{K} \notin \mathcal{K}_1$. So there is a sequent $A_0 \Rightarrow B_0 \in \Gamma_1$ such that $\alpha \not\Vdash A_0 \Rightarrow B_0$. Let $C_0 \Rightarrow D_0 \in \Gamma_2$, such that $\beta \Vdash C_0$ for some node β. It suffices to show that $\beta \Vdash D_0$. There are four cases. First, suppose $\alpha \not\Vdash A_0 \to B_0$ and $\alpha \prec \beta$. Then $\alpha \Vdash C_0 \to D_0$, hence $\beta \Vdash C_0 \Rightarrow D_0$. Second, suppose $\alpha \not\Vdash A_0 \to B_0$ and $\alpha = \beta$. Then $\alpha = \beta \Vdash D_0 \vee (A_0 \to B_0)$, so $\beta \Vdash D_0$. Third, suppose $\alpha \Vdash A_0 \to B_0$ and $\alpha \prec \beta$. Then $\alpha \Vdash A_0$, $\alpha \not\Vdash B_0$, and $\alpha \Vdash B_0 \vee (C_0 \to D_0)$. So $\alpha \Vdash C_0 \to D_0$, and $\beta \Vdash C_0 \Rightarrow D_0$. Fourth, suppose

$\alpha \Vdash A_0 \to B_0$ and $\alpha = \beta$. Then $\alpha \Vdash A_0$, $\alpha \nVdash B_0$, and $\alpha \Vdash B_0 \vee (C \to D)$ for all $C \Rightarrow D \in \Gamma_2$. So $\alpha \Vdash C \to D$ for all $C \Rightarrow D \in \Gamma_2$. Suppose $\alpha = \beta \nVdash D_0$. Then, similarly, $\alpha \Vdash A \to B$, for all $A \Rightarrow B \in \Gamma_1$. So if $\gamma \succ \alpha$, then $\gamma \Vdash \Gamma_1 \cup \Gamma_2$, thus also $\gamma \Vdash \Delta^{(1)}$. Let $E \to F \in \Delta$. Then $\gamma \Vdash E \Rightarrow F$, for all $\gamma \succ \alpha$. So $\alpha \Vdash E \to F$. Since Δ is purely formal, this implies that the root $\alpha \Vdash \Delta \supseteq \Gamma_2$, so $\mathbf{K} \in \mathcal{K}_2$. $\dashv$

Theorem 5.6 immediately implies:

Theorem 5.7 *Let Γ_1 and Γ_2 be axiomatizations of intermediate logics such that $\Gamma_1 \cup \Gamma_2$ is inconsistent. Then $Cl(\Gamma_1) \cap Cl(\Gamma_2)$ is axiomatizable by the collection of axioms*

$$A \Rightarrow B \vee (C \to D);$$

$$C \Rightarrow D \vee (A \to B); \quad \textit{and}$$

$$(A \to B) \vee (C \to D),$$

where $A \Rightarrow B$ ranges over Γ_1 and $C \Rightarrow D$ ranges over Γ_2.

Proof. Either $\Gamma_1 \subseteq E_1$ or $\Gamma_2 \subseteq E_1$. $\dashv$

Example: The intermediate logic IPC $\cap$ FPC is axiomatizable by the schemas

$$\top \to A \Rightarrow A \vee (((\top \to B) \to B) \to (\top \to B));$$

$$(\top \to B) \to B \Rightarrow (\top \to B) \vee ((\top \to A) \to A); \ \text{ and}$$

$$((\top \to A) \to A) \vee (((\top \to B) \to B) \to (\top \to B)).$$

Proposition 5.8 *The intermediate logic IPC$\cap$FPC is axiomatizable by the schemas*

$$\top \to A \Rightarrow A \vee (((\top \to B) \to B) \to (\top \to B)) \ \textit{ and}$$

$$((\top \to A) \to A) \vee (((\top \to B) \to B) \to (\top \to B)).$$

Either schema axiomatizes a proper subsystem of IPC $\cap$ FPC.

Proof. The three schemas above the Proposition axiomatize IPC $\cap$ FPC, so it suffices to show that the second schema follows from the third. The third schema obviously implies

$$(\top \to B) \to B \Rightarrow (((\top \to B) \to B) \to (\top \to B)) \vee ((\top \to A) \to A),$$

which is clearly equivalent to

$$(\top \to B) \to B \Rightarrow (\top \to (\top \to B)) \vee ((\top \to A) \to A).$$

But Formalized Transitivity gives us

$$((\top \to B) \to B) \wedge (\top \to (\top \to B)) \Rightarrow \top \to B,$$

so the second schema follows. As to the independence of the two leftover schemas, the first schema holds in all Kripke models with two nodes $\alpha \prec \beta$ and only α reflexive, and the last schema holds in all Kripke models with two nodes $\alpha \prec \beta$ and only β reflexive. But with properly chosen A and B, the reverse claims fail. $\dashv$

Another example: Let M be the theory axiomatized by the axiom $\Xi = \Xi_\perp = ((\top \to \perp) \to \perp) \to (\top \to \perp)$. BPC satisfies $\Xi \Leftrightarrow \top \to \Xi$, so M is faithful. The class of Kripke models of M consists of exactly those for which each node either equals or is below an end node. By Theorem 5.7, CPC $\cap M$ is axiomatizable by the schemas

$$\Xi \vee ((\top \to A) \to A);$$
$$\Xi \vee (\top \to (A \vee \neg A));$$
$$\top \to A \Rightarrow A \vee (\top \to \Xi);$$
$$A \vee \neg A \vee (\top \to \Xi);$$
$$(\top \to \Xi) \vee ((\top \to A) \to A); \text{ and}$$
$$(\top \to \Xi) \vee (\top \to (A \vee \neg A)).$$

Proposition 5.9 *CPC* $\cap M$ *is axiomatizable by the schemas*

$$\Xi \vee ((\top \to A) \to A); \textit{ and}$$
$$\Xi \vee A \vee \neg A.$$

Proof. Straightforward: Use that BPC satisfies $\Xi \Leftrightarrow \top \to \Xi$. The only less trivial case is deriving the third schema in the axiomatization above. From $\Xi \vee A \vee \neg A$ we get $\top \to A \Rightarrow A \vee (\top \to \bot) \vee \Xi$; but BPC satisfies $\top \to \bot \Rightarrow \Xi$. $\dashv$

Similarly, one can show

Proposition 5.10 *IPC* $\cap M$ *is axiomatizable by the schemas*

$$\Xi \vee ((\top \to A) \to A); \textit{ and}$$
$$\top \to A \Rightarrow \Xi \vee A.$$

The sequent theory CPC $\cap E_1$ is the largest intermediate logic that is consistent with all intermediate logics.

Proposition 5.11 *The intermediate logic CPC* $\cap E_1$ *is axiomatizable by the schema*

$$A \vee \neg A.$$

Proof. Obviously, the schema $A \vee \neg A$ is contained in CPC $\cap E_1$. By Theorem 5.7, CPC $\cap E_1$ is axiomatizable by the collection of schemas

$$\top \to A \Rightarrow A \vee (\top \to (\top \to \bot));$$
$$A \vee \neg A \vee (\top \to (\top \to \bot));$$
$$(\top \to \bot) \vee ((\top \to A) \to A);$$
$$(\top \to \bot) \vee (\top \to (A \vee \neg A));$$
$$((\top \to A) \to A) \vee (\top \to (\top \to \bot)); \text{ and}$$
$$(\top \to (A \vee \neg A)) \vee (\top \to (\top \to \bot)).$$

The second, fourth, and sixth schema trivially follow from the schema $A \vee \neg A$. The first schema easily follows from $(\top \to A) \wedge \neg A \Rightarrow \top \to \bot$, and the fifth schema easily follows from the third. So it suffices to derive the third schema. By the Monotonicity Proposition 2.6, we have $\top \to A \Rightarrow (\top \to A) \to A$ and $\neg(\top \to A) \Rightarrow (\top \to A) \to A$. So BPC satisfies $(\top \to A) \vee \neg(\top \to A) \Rightarrow (\top \to A) \to A$. $\dashv$

Here is a different proof of Proposition 5.11 based on Theorems 4.15 and 4.16, and the minimal model construction of Section 4.

Proposition 5.12 *$CPC \cap E_1$ is strongly complete with respect to the class of singleton models, and axiomatizable by the schema $A \vee \neg A$.*

Proof. Clearly, by Theorems 4.15 and 4.16, CPC $\cap E_1$ is weakly complete with respect to the class of singleton models. So it suffices to show that the theory C axiomatized by the schema $A \vee \neg A$ is strongly complete with respect to this class of one-node Kripke models. Obviously, all one-node models are models of C. Conversely, let $\mathbf{K}$ be a rooted model of C with root α_0 such that $\mathbf{K} \not\models \gamma$ for some sequent γ. Let $\mathbf{K}_L$ be the minimal model of $\mathbf{K}$, and suppose that $[\alpha_0] \prec_L [\beta]$. If $[\alpha] \neq [\beta]$ then, by Lemma 4.8, there exists a formula A such that $[\beta] \Vdash_L A$ and $[\alpha] \not\Vdash_L A$. So $[\alpha] \not\Vdash A \vee \neg A$, contradicting Proposition 4.5. Thus $[\alpha_0] = [\beta]$, and $\mathbf{K}_L$ is a one-node model. Apply Lemma 4.9. $\dashv$

Note that CPC $\cap E_1$ is not faithful, for it satisfies the schema $(\top \to A) \to A$, but not the sequent $\top \to \bot \Rightarrow \bot$.

A sequent $A \Rightarrow B$ is derivable from CPC $\cap E_1$, exactly when it is a classical tautology and, moreover, when after replacing all implication subformulas by $\top$, the resulting sequent $A' \Rightarrow B'$ is also a tautology. Although CPC $\cap E_1$ is a proper subsystem of CPC, they are identical in the following remarkable way:

Proposition 5.13 *Let Γ be an intermediate logic. Then for all formulas A we have $\Gamma \cap E_1 \vdash A$ if and only if $\Gamma \vdash A$.*

Proof. By Theorem 5.2 we may assume that $\Gamma \subseteq$ CPC. Suppose $\Gamma \cap E_1 \not\vdash A$, for some formula A. Then there is a Kripke model $\mathbf{K}$ such that $\mathbf{K} \not\models A$. If $\mathbf{K}$ is a model of Γ, then $\Gamma \not\vdash A$. Otherwise, we may assume that $\mathbf{K}$ is a model of E_1, hence even a singleton irreflexive model. Replace all implication subformulas in A by $\top$. The resulting formula A' is geometric and, relative to A, such that certain subformulas of A in positive places are replaced by $\top$. So BPC $\vdash A \Rightarrow A'$ and $\mathbf{K} \models A \Leftrightarrow A'$. Then BPC $\not\vdash A'$ so, by Proposition 3.8, CPC $\not\vdash A'$, and thus CPC $\not\vdash A$. $\dashv$

So, in particular, the difference between CPC $\cap E_1$ and CPC is solely in the sequents $A \Rightarrow B$ with nontrivial A. Note that the proof of Proposition 5.13 only uses that $\Gamma \subseteq$ CPC or $\Gamma \subseteq E_1$. Proposition 5.13 also applies to IPC $\cap E_1$. With Theorem 5.7 one easily verifies that IPC $\cap E_1$ is axiomatizable by the schemas

$$\top \to A \Rightarrow A \vee (\top \to (\top \to \bot)) \text{ and}$$

$$(\top \to A) \to A.$$

Proposition 5.14 *There are no intermediate logics properly between $CPC \cap E_1$ and E_1, that is, $CPC \cap E_1$ is a maximal intermediate logic inside E_1.*

Proof. Let Γ be an intermediate logic such that CPC $\cap E_1 \subseteq \Gamma \subseteq E_1$. If $\Gamma \cup \{\top \to \bot \Rightarrow \bot\}$ is consistent then, by Lemma 5.1, $\Gamma \subseteq$ CPC. So we may assume that $\Gamma \cup \{\top \to \bot \Rightarrow \bot\} \vdash \bot$. So, by the Formalization Proposition 2.3, $\Gamma \vdash (\top \to \bot) \to \bot \Rightarrow \top \to \bot$. But CPC $\cap E_1 \subseteq \Gamma$ implies that also $\Gamma \vdash (\top \to \bot) \vee ((\top \to \bot) \to \bot)$. So $\Gamma \vdash \top \to \bot$, and thus $\Gamma = E_1$. $\dashv$

Proposition 5.15 *There are no intermediate logics properly between $CPC \cap E_1$ and CPC, that is, $CPC \cap E_1$ is a maximal intermediate logic inside CPC.*

Proof. Let Γ be an intermediate logic such that $\mathrm{CPC} \cap E_1 \subseteq \Gamma \subseteq \mathrm{CPC}$. If $\Gamma \nvdash \{\top \to \bot \Rightarrow \bot\}$ then, by Proposition 2.2, $\Gamma \cup \{\top \to \bot\}$ is consistent, and $\Gamma \subseteq E_1$. So we may assume that $\Gamma \vdash \top \to \bot \Rightarrow \bot$. Let A be a formula. Then $\mathrm{CPC} \cap E_1 \subseteq \Gamma$ implies that $\Gamma \cup \{\top \to A\} \vdash ((\top \to A) \wedge A) \vee ((\top \to A) \wedge (A \to \bot))$. But $(\top \to A) \wedge (A \to \bot) \vdash \top \to \bot$, so $\Gamma \cup \{\top \to A\} \vdash A$. So $\mathrm{IPC} \subseteq \Gamma$, and thus $\Gamma = \mathrm{CPC}$. $\dashv$

Over IPC the lattice of atomless formulas, modulo provability in IPC, simply looks like

$$\begin{array}{c} \top \\ \uparrow \\ \bot \end{array}$$

The lattice of atomless formulas $\Lambda(\bot)$ over BPC, discussed below, is much more complicated. To show this we need the following results.

Lemma 5.16 *Let A be a formula such that BPC $\nvdash A$. Then there is a finite rooted Kripke model $\mathbf{K}$ with reflexive root α such that $\alpha \not\Vdash A$, and $\beta \Vdash A$ for all $\beta \neq \alpha$.*

Proof. With Theorem 4.6 there is a finite rooted Kripke model $\mathbf{K}$ such that its root $\alpha \not\Vdash A$ and $\beta \Vdash A$ for all $\beta \neq \alpha$. If α is reflexive, then we are done. Suppose α is irreflexive. Form the Kripke model $\mathbf{K'}$ from $\mathbf{K}$ by only replacing the irreflexive root α by a reflexive root α'. Again, $\beta \Vdash A$ for all $\beta \neq \alpha$. For all formulas of the form $B \to C$, if $\alpha' \Vdash B \to C$, then $\alpha \Vdash B \to C$; and for all implication-free B, $\alpha' \Vdash B$ if and only if $\alpha \Vdash B$. Now A can be written as an implication-free formula with some of its atoms replaced by implication subformulas. So if $\alpha' \Vdash A$, then $\alpha \Vdash A$. Thus $\alpha' \not\Vdash A$. $\dashv$

Theorem 5.17 *Let $A[p]$ be a formula built from the binary connectives and the atom p only, and let B be a formula such that $\top \to B \Rightarrow B$ is not derivable in BPC. Then $A[p]$ is derivable in BPC, if and only if $A[B]$ is.*

Proof. Obviously, if $A[p]$ is derivable, then so is $A[B]$. Conversely, suppose that $A[p]$ is not provable in BPC. Then there is a Kripke model $\mathbf{K}$ such that $\mathbf{K} \not\models A[p]$. By Lemma 5.16 there is a (finite) rooted Kripke model $\mathbf{K}_1$ with reflexive root α_1 such that $\alpha_1 \not\Vdash B$ and $\beta \Vdash B$ for all $\beta \succ_1 \alpha_1$. With Theorem 4.6 there is also a (finite) rooted Kripke model $\mathbf{K}_2$ with root α_2 such that $\alpha_2 \not\Vdash \top \to B \Rightarrow B$, and $\beta \Vdash \top \to B \Rightarrow B$ for all $\beta \neq \alpha_2$. So α_2 must be irreflexive and $\alpha_2 \not\Vdash B$, while $\beta \Vdash B$ for all $\beta \succ_2 \alpha_2$. Now we construct a new Kripke model $\mathbf{K'}$ as follows: All nodes $\alpha \in W$ of $\mathbf{K}$ satisfying $\alpha \Vdash p$ are in W'. Additionally, if $\alpha \not\Vdash p$, then all pairs (α, α') are in W', where α is reflexive and $\alpha' \in W_1$, or α is irreflexive and $\alpha' \in W_2$. We set $(\alpha, \alpha') \prec' (\beta, \beta')$ when $\alpha \prec \beta$ and α' is the root of its model $\mathbf{K}_i$, or $\alpha = \beta$ and $\alpha' \prec_i \beta'$ in their model $\mathbf{K}_i$; we set $(\alpha, \alpha') \prec' \beta$ if $\alpha \prec \beta$ and α' is the root of its model $\mathbf{K}_i$; and we set $\alpha \prec' \beta$ when $\alpha \prec \beta$ in $\mathbf{K}$. We set $\alpha \Vdash' q$ for all atoms q, and $(\alpha, \alpha') \Vdash' q$ exactly when $\alpha' \Vdash_i q$. Informally, we have replaced each node α of W such that $\alpha \not\Vdash p$ by a copy of $\mathbf{K}_1$ or $\mathbf{K}_2$ depending on whether α is reflexive or irreflexive. The order on the underlying W doesn't change, and the nodes in the multiple models $\mathbf{K}_i$ that are not roots are only compatible with other nodes as they were in the original $\mathbf{K}_i$. Claim: For all formulas $C[p]$ built from p with the binary connectives, and all nodes $\alpha \in W$, we have $\alpha \Vdash' C[B]$ if and only if $\alpha \Vdash C[p]$, or $(\alpha, \alpha_i) \Vdash' C[B]$ if and only if $\alpha \Vdash C[p]$, depending on whether $\alpha \in W'$ or $(\alpha, \alpha_i) \in W'$ with α_i root of $\mathbf{K}_i$. Note that if $\alpha \Vdash p$, then $\alpha \Vdash C[p]$ for all $C[p]$. Similarly, if $\alpha \Vdash' q$ for all atoms q, then $\alpha \Vdash' C[B]$ for all

$C[p]$. The proof of the claim is by a simple induction on the complexity of $C[p]$. We only show the induction step for the case where $C[p]$ is an implication $D[p] \to E[p]$. If $\alpha \not\Vdash D[p] \to E[p]$, then there is $\beta \succ \alpha$ such that $\beta \Vdash D[p]$ and $\beta \not\Vdash E[p]$. So $\beta \not\Vdash p$. By induction, $(\beta, \beta_j) \Vdash{}' D[B]$ and $(\beta, \beta_j) \not\Vdash{}' E[B]$, while $(\beta, \beta_j.) \succ' (\alpha, \alpha_i)$. So $(\alpha, \alpha_i) \not\Vdash{}' D[B] \to E[B]$. Conversely, suppose $(\alpha, \alpha_i) \not\Vdash{}' D[B] \to E[B]$. Now if β' is not a root, then $(\beta, \beta') \Vdash{}' C[B]$ for all $C[p]$; similarly, $\beta \Vdash{}' C[B]$ for all $\beta \in W' \cap W$ and $C[p]$. So there can only be a node $(\beta, \beta_i) \succ' (\alpha, \alpha_i)$ with β_i root of $\mathbf{K}_i$, such that $(\beta, \beta_i) \Vdash{}' D[B]$ and $(\beta, \beta_i) \not\Vdash{}' E[B]$. Apply induction: $\beta \succ \alpha$ is such that $\beta \Vdash D[p]$ and $\beta \not\Vdash E[p]$. So $\alpha \not\Vdash D[p] \to E[p]$. $\dashv$

In Theorem 5.17 we may add $\top$ as possible building block of $A[p]$ since, up to provable equivalence, it already occurs as $p \to p$. If $\top \to B \Rightarrow B$ holds, then the collection of formulas $A[B]$, up to provable equivalence, is very limited:

Proposition 5.18 *Let $A[p]$ be a formula built from the binary connectives, $\top$, and the atom p only, and let B be a formula such that $\top \to B \Rightarrow B$ is derivable in BPC. Then $A[B]$ is derivable in BPC, or $A[B] \Leftrightarrow B$ is derivable in BPC.*

Proof. By a simple induction on the complexity of $A[p]$. The only nontrivial case is $\top \to p$, and in that case $\top \to B \Leftrightarrow B$ is derivable in BPC. $\dashv$

The dual result of Lemma 5.16 with irreflexive roots instead of reflexive roots doesn't work as Proposition 5.19 below shows.

Proposition 5.19 *Let α be a node of a Kripke model $\mathbf{K}$, and let A be a formula, such that $\top \to A \Rightarrow A$ is derivable in BPC. If $\alpha \not\Vdash A$, then there is a node $\beta \succ \alpha$ such that $\beta \not\Vdash A$.*

Proof. If all $\beta \succ \alpha$ are such that $\beta \Vdash A$, then $\alpha \Vdash \top A$. And thus $\alpha \Vdash A$. $\dashv$

By Proposition 2.11, Proposition 5.19 applies exactly to all formulas A that are, up to provable equivalence, of the form $((\top \to B) \to B) \to (\top \to B)$.

A lattice of formulas up to provable equivalence is a basic algebra. basic algebras, defined below, are for BPC what Heyting algebras are for IPC and Boolean algebras are for CPC. A *basic algebra* $\mathcal{A}$ is a structure with constants 0 and 1, and binary functions $\wedge$, $\vee$, and $\to$, such that

- with respect to 0, 1, $\wedge$, and $\vee$ we have a distributive lattice with top and bottom; and
- for $\to$ we have the additional equations

$$a \to b \wedge c = (a \to b) \wedge (a \to c);$$

$$b \vee c \to a = (b \to a) \wedge (c \to a);$$

$$a \to a = 1;$$

$$a \leq 1 \to a; \quad \text{and}$$

$$(a \to b) \wedge (b \to c) \leq a \to c.$$

The relation $\leq$ is expressible in terms of equations with $\wedge$ in the standard way. So basic algebras form a universal algebra class with morphisms defined as usual.

Let A be a formula, and let $\Lambda(A)$ be the lattice of formulas, up to provable equivalence, generated from A using the binary connectives. So the lattice of atomless formulas is the same as $\Lambda(\bot)$. The lattices $\Lambda(A)$ are basic algebras with minimum

$A = 0$. Suppose that $\top \to A \Rightarrow A$ is not derivable. Then, by Theorem 5.17, the lattice $\Lambda(A)$ is isomorphic to the basic algebra $\Lambda(p)$. Since $\top \to \bot \Rightarrow \bot$ is not derivable, this includes the free basic algebra (on zero generators) $\Lambda(\bot)$. If, however, $\top \to A \Rightarrow A$ is derivable, but A itself is not, then $\Lambda(A)$ is the two-element lattice $\{A, \top\}$. Finally, if A is derivable as well, then $\Lambda(A)$ is the one-node basic algebra.

Let $A \in \Lambda(\bot)$. Then the assignment $\bot \mapsto A$ extends uniquely to a basic algebra endofunction on $\Lambda(\bot)$ that commutes with the binary operators and preserves 1. If A is such that $\top A \Rightarrow A$ is not derivable in BPC, then this map is an embedding. There are many such A, so $\Lambda(\bot)$ has lots of self-embeddings, hence is structurally very rich.

To get some idea of the structure of $\Lambda(\bot)$, we derive:

Proposition 5.20 *BPC satisfies the following schemas, for all $m, n \geq 0$:*

(i) $A \Rightarrow \top^m A$;

(ii) $\top^{m+1} A \to A \Leftrightarrow \top A \to A$;

(iii) $\top A \to A \Rightarrow \top^m A \to \top^n A$;

(iv) $\top A \Rightarrow \top A \to A$;

(v) $\top^{m+1} A \wedge (\top A \to A) \Leftrightarrow \top A$;

(vi) $\top^{m+2} A \to (\top A \to A) \Leftrightarrow \top^2 A \to \top A$;

(vii) $(\top^{m+1} A \to \top^m A) \to A \Leftrightarrow \top A$;

(viii) $(\top A \to A) \to \top^{m+1} A \Leftrightarrow (\top A \to A) \to \top A$; *and*

(ix) $(\top^{m+1} A \to \top^m A) \to \top^{m+1} A \Rightarrow (\top A \to A) \to \top A$.

Proof. (i) follows from the schema $A \Rightarrow \top \to A$.

As to (ii): The implication $\Rightarrow$ follows from the Monotonicity Proposition 2.6; the reverse implication $\Leftarrow$ follows from $\top A \to A$ being equivalent to $\top A \leftrightarrow A$, and repeated application of the Formal Substitution Proposition 2.5.

(iii) immediately follows from (ii).

(iv) follows from $\top A \Leftrightarrow \top \to A$ and the Monotonicity Proposition.

As to (v): The $\Leftarrow$ direction immediately follows from (i) and (iv). The converse direction follows by induction on m: $\top^{m+1} A \wedge (\top A \to A) \Leftrightarrow (\top \to \top^m A) \wedge (\top \to (\top A \to A)) \wedge (\top A \to A) \Leftrightarrow (\top \to (\top^m A \wedge (\top A \to A)) \wedge (\top A \to A) \Leftrightarrow (\top \to \top A) \wedge (\top A \to A) \Leftrightarrow \top A$.

As to (vi): Suppose $\top^{m+2} A \to (\top A \to A)$. Then, with (ii), we get $\top^{m+2} A \to ((\top \to \top^{m+1} A) \wedge (\top^{m+1} A \to A))$, hence $\top^{m+2} A \to \top A$, and thus $\top^2 A \to \top A$. Conversely, $\top^2 A \to \top A$ is the same as $\top^{m+2} A \to \top A$, which is the same as $\top^{m+2} A \to (\top \to A)$: Apply the Monotonicity Proposition.

As to (vii): The direction $\Leftarrow$ immediately follows from the Monotonicity Proposition. Conversely, with (iv), we get $(\top^{m+1} A \to \top^m A) \to A \Rightarrow \top^{m+1} A \to A \Rightarrow \top^{m+1} A \to \top^m A$; and thus $(\top^{m+1} A \to \top^m A) \to A \Rightarrow \top \to A$.

As to (viii): The direction $\Leftarrow$ immediately follows from the Monotonicity Proposition. Conversely, with (ii), we get $(\top A \to A) \to \top^{m+1} A \Leftrightarrow (\top A \to A) \to ((\top \to \top^m A) \wedge (\top^m A \to A)) \Rightarrow (\top A \to A) \to \top A$.

As to (ix): By (iii) and the Monotonicity Proposition we have $(\top^{m+1} A \to \top^m A) \to \top^{m+1} A \Rightarrow (\top A \to A) \to \top^{m+1} A$. Apply(viii). $\dashv$

References

[1] M. Ardeshir. *Aspects of Basic Logic.* PhD Thesis, Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, 1995.

[2] H. Barendregt, M. Bezem, J.W. Klop (editors). *Dirk van Dalen Festschrift.* Quaestiones Infinitae, Vol. 5. Department of Philosophy, Utrecht University, 1993.

[3] G. Boolos. *The Unprovability of Consistency: An Essay in Modal Logic.* Cambridge University Press, 1979.

[4] S.A. Kripke. *A completeness theorem in modal logic.* The Journal of Symbolic Logic **24**, 1959, 1–14.

[5] D.C. Makinson. *Some embedding theorems for modal logic.* Notre Dame Journal of Formal Logic **12** 1971, 252-254.

[6] W. Ruitenburg. *Constructive logic and the paradoxes.* Modern Logic **1**, 1991, 271–301.

[7] W. Ruitenburg. *Basic Logic and Fregean Set Theory.* In: [2], 121–142.

[8] K. Segerberg. *Modal logics with linear alternative relations.* Theoria **36**, 1970, 301–322.

[9] C. Smoryński. *Self-Reference and Modal Logic.* Universitext. Springer, 1985.

[10] A. Visser. *A propositional logic with explicit fixed points.* Studia Logica **40**, 1981, 155–175.